Categorification of $U_q(sl_2)$: Part III

Let χ = set of invertible elements in k, β_n $(n \in \mathbb{Z})$, c_n^+ $(n \in \mathbb{Z}^+)$, c_n^- $(n \in \mathbb{Z}^-)$,

Def.: 2-category $\mathcal{U}_\chi$

- Objects: $n \in \mathbb{Z}$
- 1-morphisms: $\mathcal{E}_{(\varepsilon_1)} 1_n \{t_1\} \oplus \dots \oplus \mathcal{E}_{(\varepsilon_N)} 1_n \{t_N\} \in 1\mathrm{Morph}(n, m)$
  $\hookrightarrow$ $(\varepsilon) = (\varepsilon_1, \dots, \varepsilon_m)$, $\varepsilon_i \in \{+, -\}$, $\mathcal{E}_{(\varepsilon)} = \mathcal{E}_{\varepsilon_1} \cdots \mathcal{E}_{\varepsilon_m}$
  $\mathcal{E}_+ = \mathcal{E}$, $\mathcal{E}_- = \mathcal{F}$, $m = n + 2\sum_i \varepsilon_i \cdot 1$
  Composition: $\mathcal{E}_{(\varepsilon')} 1_m \{t'\} \circ \mathcal{E}_{(\varepsilon)} 1_n \{t\} = \mathcal{E}_{(\varepsilon' \cup \varepsilon)} 1_n \{t + t'\}$
- $2\text{-}\mathrm{Morph}_{\mathcal{U}_\chi}(\mathcal{E}_{(\varepsilon)} 1_n \{t\}, \mathcal{E}_{(\varepsilon')} 1_n \{t'\}) = k$-vector space of lin combi of diagrams $\rightsquigarrow$ horizontal / vertical composition of generating 2-morphisms of degree $t - t'$

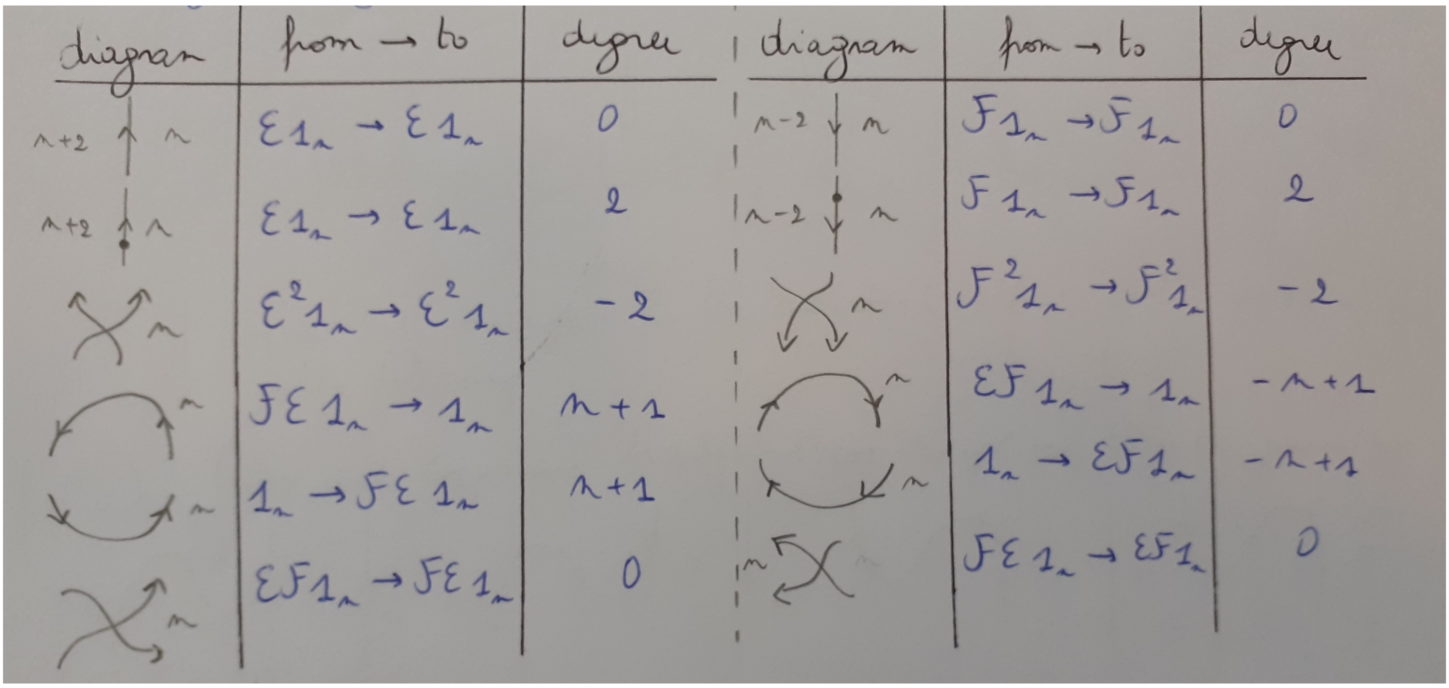
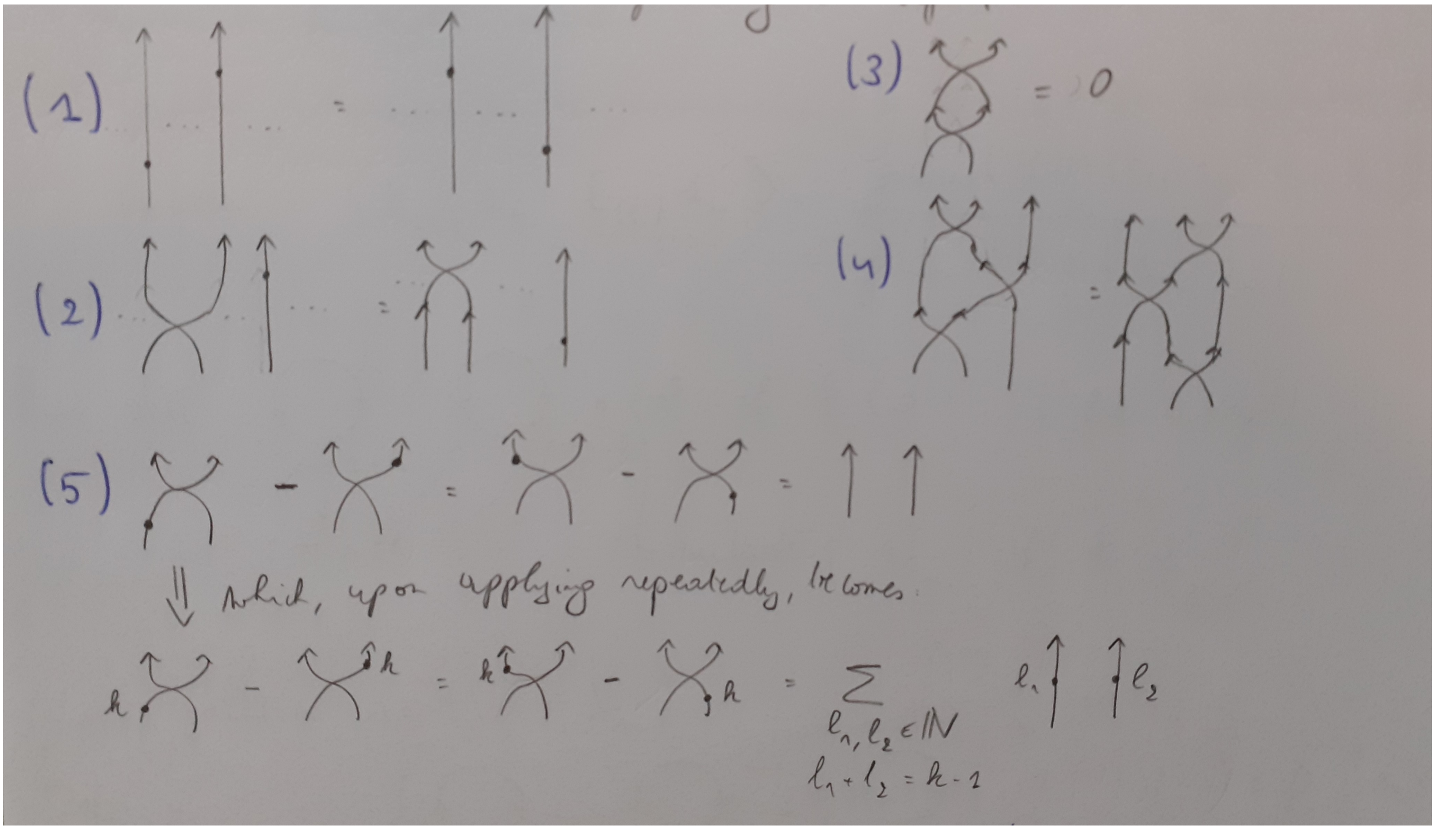


diagram	from → to	degree	diagram	from → to	degree
$n+2$ ↑ n	$\mathcal{E}1_n \to \mathcal{E}1_n$	0	$n-2$ ↓ n	$\mathcal{F}1_n \to \mathcal{F}1_n$	0
$n+2$ ↑• n	$\mathcal{E}1_n \to \mathcal{E}1_n$	2	$n-2$ ↓• n	$\mathcal{F}1_n \to \mathcal{F}1_n$	2
n	$\mathcal{E}^2 1_n \to \mathcal{E}^2 1_n$	-2	n	$\mathcal{F}^2 1_n \to \mathcal{F}^2 1_n$	-2
n	$\mathcal{F}\mathcal{E}1_n \to 1_n$	$n+1$	n	$\mathcal{E}\mathcal{F}1_n \to 1_n$	$-n+1$
n	$1_n \to \mathcal{F}\mathcal{E}1_n$	$n+1$	n	$1_n \to \mathcal{E}\mathcal{F}1_n$	$-n+1$
n	$\mathcal{E}\mathcal{F}1_n \to \mathcal{F}\mathcal{E}1_n$	0	n	$\mathcal{F}\mathcal{E}1_n \to \mathcal{E}\mathcal{F}1_n$	0

which satisfy relations:

(1) Nil-Hecke relations:

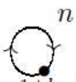
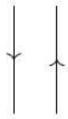
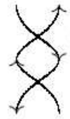
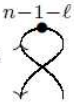


(2) The rel. that guarantee that

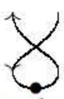
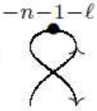
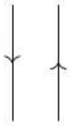
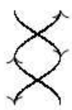
$\sigma_+ := \oplus \underset{n-1}{} \oplus \cdots \oplus \, : \mathcal{F}\mathcal{E}1_n \oplus 1_n\{n-1\} \oplus \cdots \oplus 1_n\{-n+1\} \to \mathcal{E}\mathcal{F}1_n$

is isomorphism, with inverse

$\overline{\sigma_+} := \beta_n \; {}^{n} \oplus \bigoplus_{\ell=0}^{n-1} \bigoplus_{\substack{\lambda \text{ partition} \\ |\lambda| \le \ell}} \alpha_\lambda^\ell(n) \; C_{\lambda,n} \quad \ell-|\lambda| \quad \underset{n-1+\lambda_1}{} \cdots \underset{n-1+\lambda_m}{}$ $\qquad \lambda = (\lambda_1, \dots, \lambda_m)$

Relations for $n \geq 0$	
(A1)	$\delta_{b,0} = \sum_{\lambda:\lvert\lambda\rvert\leq b} \alpha_\lambda^\ell(n)\, e_{\lambda,n}$  n $n-1+b-\lvert\lambda\rvert$
(A2)	n  n $=$ β_n  n
(A3)	β_n  $n-1-\ell$ n $= 0$
(A4)	$\sum_\lambda \alpha_\lambda^\ell(n)\, e_{\lambda,n}$ $n-1-\ell$  n $= 0$
(A5)	n  n $= \beta_n$  n $+ \sum_{\substack{f_1+f_2+\lvert\lambda\rvert \\ =n-1}} \alpha_\lambda^{\lvert\lambda\rvert+f_2}(n) e_{\lambda,n}$ n  f_1 f_2 

Note that relations **A1**, **A3**, and **A4** are only valid for $n > 0$.

Relations for $n \geq 0$	
(B1)	$\delta_{b,0} = \sum_{\lambda:\lvert\lambda\rvert\leq b} \alpha_\lambda^\ell(n)\, e_{\lambda,n}$  n $-n-1+b-\lvert\lambda\rvert$
(B2)	n  n $=$ β_n  n
(B3)	β_n  $-n-1-\ell$ n $= 0$
(B4)	$\sum_\lambda \underline{\alpha_\lambda^\ell(n)}\, e_{\lambda,n}$ $-n-1-\ell$  n $= 0$
(B5)	n  n $= \underline{\beta_n}$  n $+ \sum_{\substack{g_1+g_2+\lvert\lambda\rvert \\ =-n-1}} \alpha_\lambda^{\lvert\lambda\rvert+g_2}(n)\, e_{\lambda,n}$ n  g_1 g_2 

Note that relations **B1**, **B3**, and **B4** are only valid for $n < 0$.

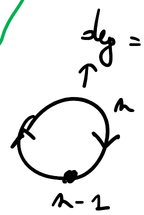
The map $\lambda \mapsto e_{\lambda,n}$ is injective $\Rightarrow$ $\alpha_\lambda^\ell(n)$ are completely determined by (A1) - (B1)

$\Updownarrow$

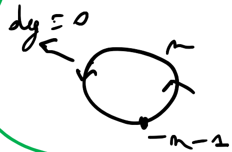
$$\left.\sum_{\substack{\lambda \text{ partition} \\ |\lambda| \leq b}} \alpha_\lambda^\ell(n)\, e_{\lambda,n}\, e_{b-|\lambda|,n} = \delta_{b,0} \quad \forall b \in \mathbb{N}\right] \ (*)$$

$\hookrightarrow$ Remark:

Dotted bubbles of degree 0 = multiple of 1_{1_n}:



$= c_n^+ 1_{1_n}$ if $n \geq 1$



$= c_n^- 1_{1_n}$ if $n \leq -1$

$$\Rightarrow \alpha_\lambda^\ell(n) = \frac{\alpha_\lambda^\ell(n)}{(c_n^\pm)^{m+1}}$$

$\hookrightarrow (\lambda_1, \dots, \lambda_m)$

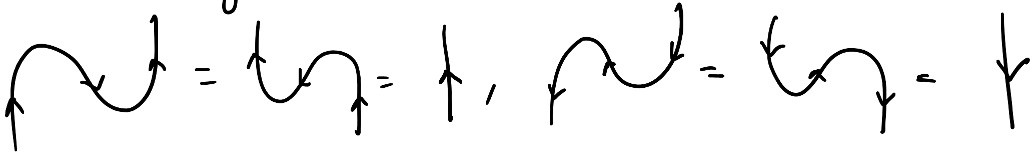
Coeff that expresses a complete symm. function in terms of elementary symm. functions

$$\hookrightarrow h_{s,n} := (-1)^s \sum_{|\lambda| = s} \alpha_\lambda^\ell(n)\, e_{\lambda,n}$$

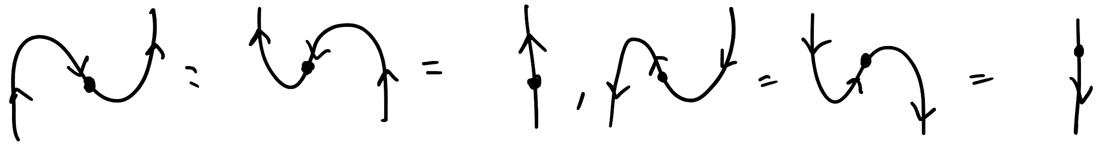
$$(*) \Leftrightarrow \sum_{\substack{s, r \in \mathbb{N} \\ s+r=b}} (-1)^s\, h_{s,n}\, e_{r,n} = \delta_{b,0}. \quad \forall b \in \mathbb{N}$$

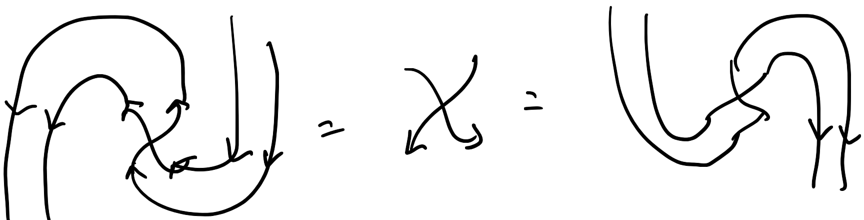
Equiv. relations for 2-morphisms:

(2'a): Biadjointness:



(2'b): Cyclicity w.r.t. biadjointness structure:

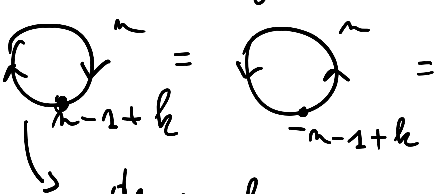




+ some others

$\Rightarrow$ Isotopic diagrams = same 2-morphism

(2'c): • Bubbles of degree < 0 should vanish:

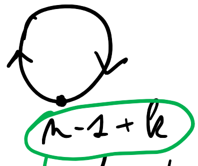


$= 0$ if $k < 0$

degree $= k$

• Bubbles of degree $0 = \text{cte}.\ 1_{1_n}$

• Fake bubbles: positive degree BUT carry a negative number of dots

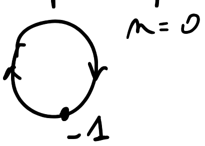


$n < 0$
$0 \leq k < -n$ $\Rightarrow$ not allowed by graphical calculus

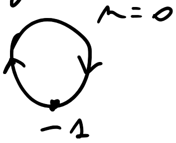
↳ degree $= k \geq 0$

< 0

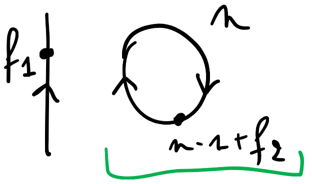
$\Rightarrow$ Interpret fake bubbles as formal symbols:

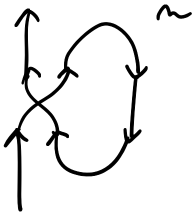
(A) 

$= c_0^+ \, 1_{1_0}$,

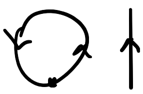


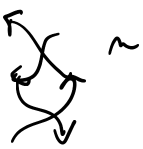
$= c_0^- \, 1_{1_0}$

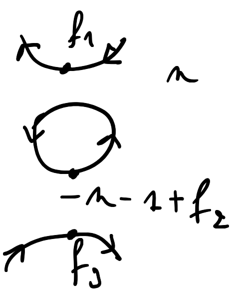
(B) $\sum_{\substack{f_1, f_2 \in \mathbb{N} \\ f_1 + f_2 = -n}}$ 

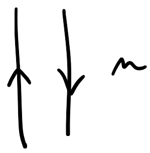
$= -$ 

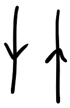
$\forall n \in \mathbb{Z}$

Similar with 

(C) β_n 

$- \beta_n \sum_{\substack{f_1, f_2, f_3 \in \mathbb{N} \\ f_1 + f_2 + f_3 = n-1}}$ 

$=$ 

+ similar 

Prop.: Set of coeff X is completely determined by β_n, ($n \in \mathbb{Z}$) and c_0^+, c_1^+:

- $c_0^+ c_0^- \beta_0 = -1$
- $c_1^+ c_{-1}^- = 1$
- $c_{n-2}^+ = -\beta_n c_n^+$ if $n \geq 0$
- $c_{n-2}^- = -\beta_n c_n^-$ if $n \leq 0$

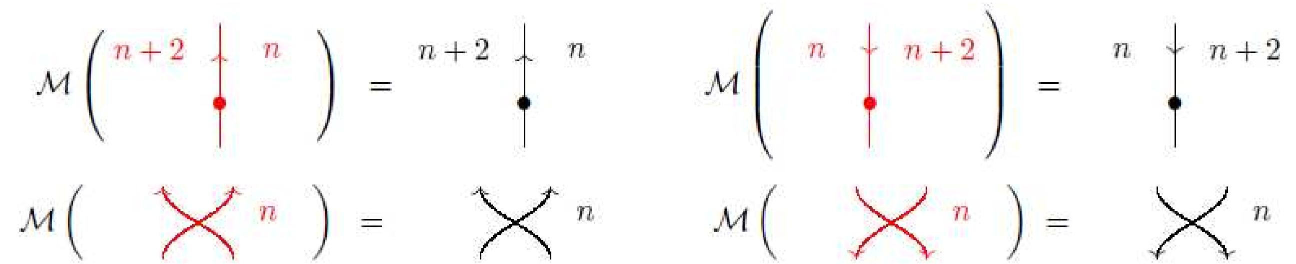
One can always choose: $\beta_n = -1$, $c_n^+ = 1$, $c_n^- = 1$ $\forall n$

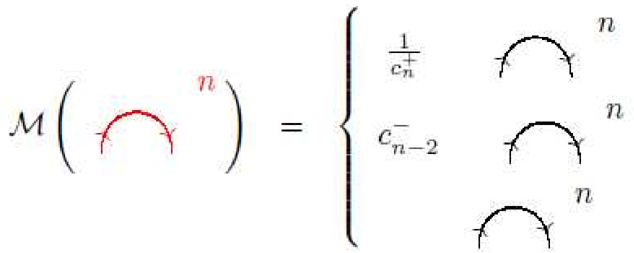
Def. : $\mathcal{U} := \mathcal{U}_\chi$ with $\chi = \{\beta_n = -1,\ c_n^{\pm} = 1\}_{n \in \mathbb{Z}}$

Theorem: Let χ be any set of invertible scalars that satisfy relations of Prop.

$\mathcal{M}$: natural transformation $\mathcal{U} \to \mathcal{U}_\chi$:

$$n \in Ob(\mathcal{U}) \mapsto n$$
$$\mathcal{E}_{(\varepsilon)} 1_n\{t\} \mapsto \mathcal{E}_{(\varepsilon)} 1_n\{t\}$$

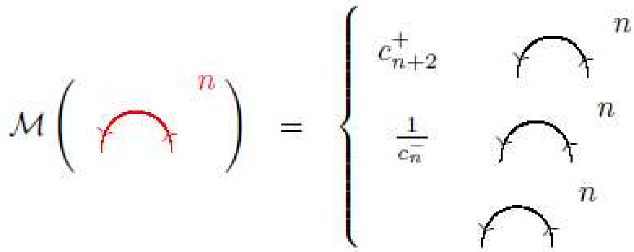




$n = 2\ell$, or $n = 2\ell + 1$, for $\ell \in 2\mathbb{Z}_{\geq 0}$

$n = -2\ell$, or $n = -(2\ell + 3)$, for $\ell \in 2\mathbb{Z}_{\geq 0}$

otherwise



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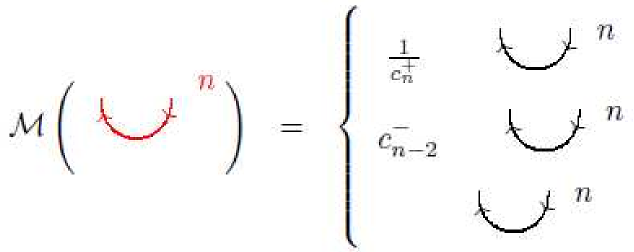
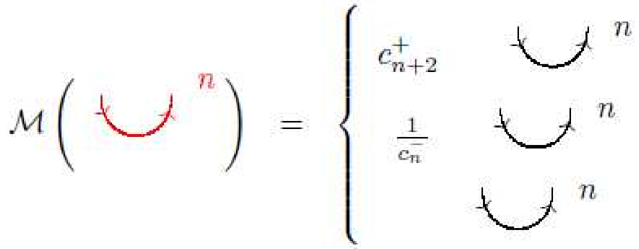


$n = 2\ell + 2$, or $n = 2\ell + 3$, for $\ell \in 2\mathbb{Z}_{\geq 0}$

$n = -(2\ell + 1)$, or $n = -(2\ell + 2)$, for $\ell \in 2\mathbb{Z}_{\geq 0}$

otherwise



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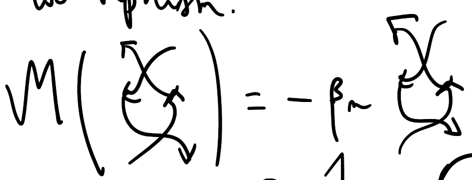
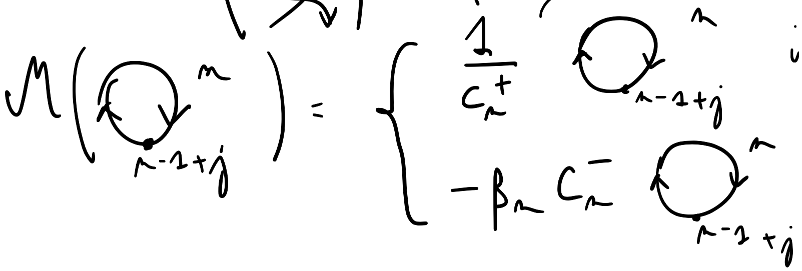
otherwise

$\mathcal{M}$ is isomorphism.

Proof:





if bubble is real

if bubble is fake

$\Rightarrow \mathcal{M}$ is 2-functor + invertible (bc. $c_n^{\pm}$, β_n are invertible). ☐

Recall: There only exists a categorification for ${}_{\mathcal{A}}\dot{U}$ (integral form of $\dot{U}$)

$\mathbb{Z}[q,q^{-1}]$-subalgebra of $\dot{U}$ generated by "divided powers

$$\frac{E^a}{[a]!}, \frac{F^b}{[b]!} \quad a,b \in \mathbb{N}, \quad [a]! = [a][a-1]\dots[1],$$

$$[a] = \frac{q^a - q^{-a}}{q - q^{-1}}$$

Why? • q-Serre rel. in $U_q(\mathfrak{g})$

have only $\mathbb{Z}[q,q^{-1}]$ coeff if we work in divided powers

• $\mathcal{E}_{(\varepsilon_1)} 1_n\{t\} \oplus \dots$

only accounts for the $\mathbb{Z}[q,q^{-1}]$-module structure of $1_m {}_{\mathcal{A}}\dot{U} 1_n$,

NOT for the $\mathbb{Q}(q)$-module structure of $1_m \dot{U} 1_n$.

$\Downarrow$

$\dot{\mathcal{U}}$ should have 1-morphisms: $\mathcal{E}_{(\varepsilon)} 1_n\{t_1\} \oplus \dots \oplus \mathcal{E}_{(\varepsilon_N)} 1_n\{t_N\}$,

where $(\varepsilon) = (\varepsilon_1^{a_1}, \dots, \varepsilon_m^{a_m})$, $a_i \in \mathbb{N}$, $\varepsilon_i \in \{+,-\}$

$$\mathcal{E}_{(\varepsilon)} = \mathcal{E}_{\varepsilon_1}^{(a_1)} \dots \mathcal{E}_{\varepsilon_m}^{(a_m)} \longrightarrow \mathcal{E}_+^{(a)} = \mathcal{E}^{(a)}, \quad \mathcal{E}_-^{(a)} = \mathcal{F}^{(a)}$$

$$\mathcal{E}^{(a)} \rightsquigarrow \frac{E^a}{[a]!}, \qquad \mathcal{F}^{(a)} \rightsquigarrow \frac{F^a}{[a]!}$$

$\mathcal{E}^a 1_n \cong \bigoplus_{[a]!} \mathcal{E}^{(a)} 1_n$

in $\mathcal{U}$ in $\dot{\mathcal{U}}$

$\mathcal{E}^{(a)} 1_n$

$\hookrightarrow$ e.g. • $a=2$: $[2]! = q + q^{-1}$

$\Rightarrow \mathcal{E}^2 1_n \cong \mathcal{E}^{(2)} 1_n\{1\} \oplus \mathcal{E}^{(2)} 1_n\{-1\}$

• $a=3$: $[3]! = q^3 + 2q + 2q^{-1} + q^{-3}$

$\Rightarrow \mathcal{E}^3 1_n \cong \mathcal{E}^{(3)} 1_n\{3\} \oplus \mathcal{E}^{(3)} 1_n\{1\} \oplus \mathcal{E}^{(3)} 1_n\{1\} \oplus \dots$

Solution: $1\mathrm{Morph}_{\dot{\mathcal{U}}}(m,n) = \mathrm{Kar}(1\mathrm{Morph}_{\mathcal{U}}(m,n))$

The Karoubi envelope.

- idempotent: morphism $e: b \to b$ s.t. $e \circ e = e$
- split idempotent: $\exists g: b \to b'$, $h: b' \to b$: $e = h \circ g$, $g \circ h = 1_{b'}$

  ↳ If $\mathcal{C}$ is additive category: $b' := \operatorname{im} e \rightsquigarrow$ $g = pr_{\operatorname{im} e}$

  $b = \operatorname{im}(e) \oplus \operatorname{im}(1-e)$ $\quad h = inj_{\operatorname{im}(e) \hookrightarrow b}$

- $Kar(\mathcal{C})$: • Objects: (b, e) $\quad b \in Ob(\mathcal{C})$, e idempotent in $\mathcal{C}$

  • Morphisms: (e, f, e') $\quad f \in Morph_{\mathcal{C}}(b, b')$

  $e' \circ f = f \circ e = f$ $\quad e: b \to b$, $e': b' \to b'$

  ↳ If $\mathcal{C}$ is additive category $\Rightarrow$ $Kar(\mathcal{C})$ is additive

  $\operatorname{im} e := (b, e) \in Ob(Kar(\mathcal{C})) \Rightarrow \underset{\in Ob(\mathcal{C})}{b} \cong \operatorname{im}(e) \oplus \underset{Ob(Kar(\mathcal{C}))}{\operatorname{im}(1-e)}$

$\mathcal{U}$ = additive 2-category

- Objects: $n \in \mathbb{Z}$
- $1\text{-}Morph_{\mathcal{U}}(n, m)$: $\left(\mathcal{E}_{(\varepsilon_1)} 1_n\{t_1\}, e_1\right) \oplus \dots \oplus \left(\mathcal{E}_{(\varepsilon_N)} 1_n\{t_N\}, e_N\right)$

  $e_i \in 2Morph_{\mathcal{U}}\left(\mathcal{E}_{(\varepsilon_i)} 1_n\{t_i\}, -\right)$

- $2\text{-}Morph_{\mathcal{U}}\left(\left(\mathcal{E}_{(\varepsilon)} 1_n\{t\}, e\right), \left(\mathcal{E}_{(\varepsilon')} 1_n\{t'\}, e'\right)\right) = (e, f, e')$

  $f \in 2Morph_{\mathcal{U}}\left(\mathcal{E}_{(\varepsilon)} 1_n\{t\}, \mathcal{E}_{(\varepsilon')} 1_n\{t'\}\right)$

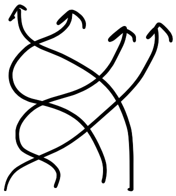
Prefer: Description with $\mathcal{E}^{(a)} 1_n$

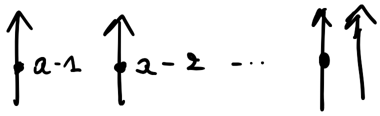
- minimal idempotent: idempotent e which cannot be written as $e' \oplus e''$, s.t. $e''e' = e'e'' = 0$

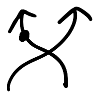
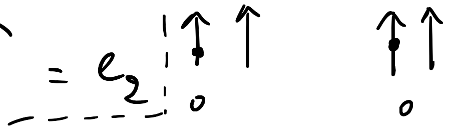
Def.: e_a minimal idempotent $\in 2Morph_{\mathcal{U}}(\mathcal{E}^a 1_n, \mathcal{E}^a 1_n)$ (up to grading shift)

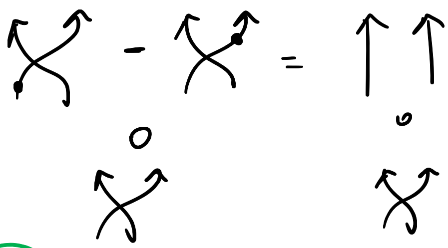
↳ Let D_a = longest braid on a strands

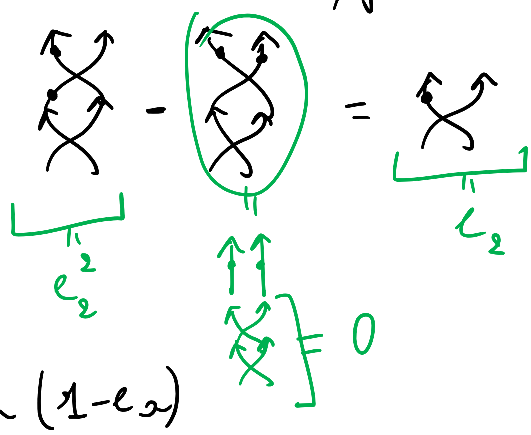
|

e.g.: $a = 4$: 

Def.: $e_a :=$  $\circ\ D_a$

↳ Exemple: $a=2$:  $= e_2$ 

↳ Idempotent? 

$\Longleftrightarrow$  $\Longleftrightarrow e_2^2 = e_2$

im $e_a \cong$ im $(1-e_a)$
(up to grading shift)

Def.: $\forall a \in \mathbb{N}$: $\mathcal{E}^{(a)} 1_n := \left(\mathcal{E}^a 1_n \left\{ \frac{-a(a-1)}{2} \right\}, e_a \right)$

$\in \mathrm{Kar}(1\mathrm{Morph}_{\mathcal{U}}(m,n))$
$= 1\mathrm{Morph}_{\dot{\mathcal{U}}}(m,n)$

$\Downarrow$

Prop.: $\mathcal{E}^a 1_n \cong \bigoplus_{[a]!} \mathcal{E}^{(a)} 1_n$.

Remark: Graphical calculus doesn't really work anymore for $\dot{\mathcal{U}}$

Categorification theorem:

- $\gamma : {}_{\mathcal{A}}\dot{U} \to K_0(\dot{\mathcal{U}})$: $E_{(\varepsilon)} 1_n \longmapsto [\mathcal{E}_{(\varepsilon)} 1_n]$

  ↳ $\varepsilon = (\varepsilon_1^{a_1}, \dots, \varepsilon_N^{a_N})$: $E^{(a)}_{(\varepsilon)} = \dfrac{E_\pm^a}{[a]!}$

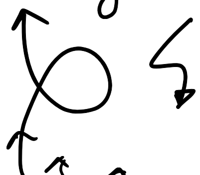
  is isomorphism of $\mathbb{Z}[q, q^{-1}]$-algebras

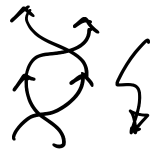
- The indecomposable 1-morphisms in $\dot{\mathcal{U}}$ (up to grading shift) give basis in $K_0(\dot{\mathcal{U}})$ with structure coeffs $\in \mathbb{N}[q,q^{-1}]$, s.t. $\gamma^{-1}(\text{basis})$ = Lusztig's canonical basis of ${}_{\mathcal{A}}\dot{U}$.

"Proof"/Motivation:

1. The spaces of 2-morphisms are not "too large"

   Each 2HOM : lin. comb of "clean diagrams"

   ↳ no self-intersections of strands: 

   ↳ no strands intersect more than once: 

   ↳ all dots $\in$ small interval 

   ↳ Closed diagrams:
   products of dotted bubbles,
   - ↳ non-nested
   - ↳ same orientation
   - ↳ far right corner

   Clean diagrams = basis ⇝ ∃ analogs of Reidemeister moves

   $$\Rightarrow \operatorname{gdim}\left(\mathrm{HOM}_{\dot{\mathcal{U}}}(\mathcal{E}_{(\varepsilon)}1_n, \mathcal{E}_{(\varepsilon')}1_n)\right) = \langle E_{(\varepsilon)}1_n, E_{(\varepsilon')}1_n\rangle \cdot \frac{1}{\prod_{a=1}^{\infty} 1-q^{2a}}$$

   $$\frac{1}{\prod_{a=1}^{\infty} 1-q^{2a}} = \operatorname{gdim}\left(\mathrm{HOM}_{\dot{\mathcal{U}}}(1_n,1_n)\right)$$

2. Relations are "not too strong"

   2-Morph $\neq \{0\}$

3. γ = homomorphism: ↗ ∃ $\varphi^{\pm}$ isomorphisms
   ↘ Composition of 1-morph ⟺ multiplication in $\dot{U}$