

Categorification of quantum sl_2 : Part II

Recall from last week:

• \dot{U} = idempotent part of $U_q(sl_2)$ = associative non-unital algebra over $\mathbb{Q}(q)$ with generators $1_m, 1_{m+2} E 1_m, 1_{m-2} F 1_m$ ($m \in \mathbb{Z}$)

and relations: $E 1_m = 1_{m+2} E$ $F 1_m = 1_{m-2} F$

$\rightarrow \cdot 1_m 1_m = \delta_{m,0} 1_0$

$\rightarrow 1_m E 1_m = 0$

$\rightarrow \cdot [E, F] 1_m = [m] 1_m = \frac{q^m - q^{-m}}{q - q^{-1}} \cdot 1_m$

if $m \neq m+2$

$\hookrightarrow = (q^{m-1} + q^{m-3} + \dots + q^{-m+1}) 1_m$

• 2-Category \mathcal{U} with: • objects: $m \in \mathbb{Z}$

• 1-Morphisms (m, m) :

$\rightarrow 1_m \varepsilon_{(\varepsilon_i)} 1_m \{t_i\} \oplus \dots \oplus 1_m \varepsilon_{(\varepsilon_N)} 1_m \{t_N\}$

\hookrightarrow generating 1-morphisms: $1_m, E, F$

$\hookrightarrow t_1, \dots, t_N \in \mathbb{Z}$

$\hookrightarrow \varepsilon = (\varepsilon_1, \dots, \varepsilon_N), \varepsilon_i \in \{+, -\}, \varepsilon_{(\varepsilon)} = \varepsilon_{\varepsilon_1} \dots \varepsilon_{\varepsilon_N}$

$\varepsilon_+ = E, \varepsilon_- = F, m = m + 2 \sum_{i=1}^N \varepsilon_i \cdot 1$

• 2-Morphisms

$\hookrightarrow K_0(1\text{Morph}_{\mathcal{U}}(m, m)) = 1_m \dot{U} 1_m, \mathbb{Z}[q, q^{-1}]$ -module

$\hookrightarrow [f\{t\}] = q^t [f]$

$\hookrightarrow \mathcal{U}$ = additive 2-category

\hookrightarrow Composition of 1-morphisms in $\mathcal{U} \Leftrightarrow$ multiplication in \dot{U}

$\mathbb{Z}[q, q^{-1}]$ -semilinear form on \dot{U}

• $\langle E^a 1_m, E^a 1_m \rangle = \langle F^a 1_m, F^a 1_m \rangle = \frac{([a]!)^2}{\prod_{j=1}^a (1 - q^{2j})}$

$[a]! = [a][a-1] \dots [1]$

• $\langle ux, y \rangle = \langle x, \tau(w)y \rangle, u, x, y \in \dot{U}$

\rightarrow with $\tau(E 1_m) = q^{-m+2} 1_m F, \tau(F 1_m) = q^{m+2} 1_m E$

Write $\mathcal{U}(f, g) = 2\text{Morph}_{\mathcal{U}}(f, g), \rightarrow \text{Hom}_{\mathcal{U}}(f, g) = \bigoplus_{t \in \mathbb{Z}} \mathcal{U}(f\{t\}, g)$

1. $g \in 1\text{Morph}(m, n)$

$\langle [f], [g] \rangle = g \dim(\text{Hom}_{\mathcal{U}}(f, g)) = \sum_{t \in \mathbb{Z}} q^t \dim(\mathcal{U}(f\{t\}, g)) \Leftarrow$

Find the vector space $\mathcal{U}(\mathcal{E}_{(\varepsilon)} 1_n \{t\}, \mathcal{E}_{(\varepsilon')} 1_n \{t'\})$, $\forall \varepsilon, \varepsilon', n, t', t$
 w.l.o.g.: $t'=0$

$$\begin{aligned} & \dim(\text{Hom}(\mathcal{E}_{(\varepsilon)} 1_n \{t\}, \mathcal{E}_{(\varepsilon')} 1_n \{t'\})) \\ &= \dim(\text{Hom}(\mathcal{E}_{(\varepsilon)} 1_n \{t-t'\}, \mathcal{E}_{(\varepsilon')} 1_n)) \end{aligned}$$

$$\begin{aligned} \text{Ansatz: } \langle \mathcal{E}_{(\varepsilon)} 1_n, \mathcal{E}_{(\varepsilon')} 1_n \rangle &= \dim(\text{Hom}(\mathcal{E}_{(\varepsilon)} 1_n, \mathcal{E}_{(\varepsilon')} 1_n)) \\ &= \sum_{t \in \mathbb{Z}} q^t \dim(\mathcal{U}(\mathcal{E}_{(\varepsilon)} 1_n \{t\}, \mathcal{E}_{(\varepsilon')} 1_n)) \end{aligned}$$

1. $\text{Hom}(\mathcal{E} 1_n, \mathcal{E} 1_n)$, $(\varepsilon) = (\varepsilon') = (+)$

$$\sum_{t \in \mathbb{Z}} q^t \dim(\mathcal{U}(\mathcal{E} 1_n \{t\}, \mathcal{E} 1_n)) = \langle \mathcal{E} 1_n, \mathcal{E} 1_n \rangle = \frac{1}{1-q^2} = 1 + q^2 + q^4 + q^6 + \dots$$

$$\Rightarrow \forall t \in \mathbb{Z}: \dim(\mathcal{U}(\mathcal{E} 1_n \{t\}, \mathcal{E} 1_n)) = \begin{cases} 0 & \text{if } t < 0 \text{ or } t \text{ is odd} \\ 1 & \text{if } t \in 2\mathbb{N} \end{cases}$$

$t=0$: $\mathcal{U}(\mathcal{E} 1_n, \mathcal{E} 1_n) = \mathbb{K}$ -linear span of $\mathbb{1}_{\mathcal{E} 1_n}$

$\begin{matrix} \xrightarrow{n+2} \\ \mathcal{E} 1_n \\ \downarrow \\ \mathcal{E} 1_n \end{matrix} \quad \begin{matrix} \uparrow n \\ \mathcal{E} 1_n \end{matrix}$

$t=2$: 2-morph $\mathcal{E} 1_n \{2\} \rightarrow \mathcal{E} 1_n$

$\deg(\begin{matrix} \uparrow n+2 \\ \mathcal{E} 1_n \end{matrix} \downarrow \begin{matrix} \mathcal{E} 1_n \\ \downarrow \\ \mathcal{E} 1_n \end{matrix}) = 2$

$t=4$: $\mathcal{E} 1_n \{4\} \rightarrow \mathcal{E} 1_n$: Vertical composition

$\deg(\begin{matrix} \uparrow n+2 \\ \mathcal{E} 1_n \end{matrix} \downarrow \begin{matrix} \mathcal{E} 1_n \\ \downarrow \\ \mathcal{E} 1_n \end{matrix}) = 4$

$\begin{matrix} \mathcal{E} 1_n & \mathcal{E} 1_n \{2\} \\ \uparrow n+2 & \uparrow n \\ \mathcal{E} 1_n \{2\} & \mathcal{E} 1_n \{4\} \end{matrix}$

$$\begin{matrix} \uparrow n+2 \\ \downarrow \mathbb{K} \\ \mathcal{E} 1_n \end{matrix} = \begin{matrix} \uparrow \mathbb{K} \\ \downarrow \\ \mathcal{E} 1_n \end{matrix}$$

$\Rightarrow \mathcal{U}(\mathcal{E} 1_n \{2k\}, \mathcal{E} 1_n) = \mathbb{K}$ -lin. span of $\begin{matrix} \uparrow n+2 \\ \downarrow \mathbb{K} \\ \mathcal{E} 1_n \end{matrix}$

$k \in \mathbb{N} \setminus \{0\}$

Similarly for $\mathcal{F} 1_n, \mathcal{F} 1_n$: $\mathcal{U}(\mathcal{F} 1_n, \mathcal{F} 1_n) = \langle \begin{matrix} \uparrow n-2 \\ \downarrow \mathbb{K} \\ \mathcal{F} 1_n \end{matrix} \rangle$

$\mathcal{U}(\mathcal{F} 1_n \{2k\}, \mathcal{F} 1_n) = \langle \begin{matrix} \uparrow n-2 \\ \downarrow \mathbb{K} \\ \mathcal{F} 1_n \end{matrix} \rangle$

2. $\text{Hom}(E\varepsilon_{1n}, E\varepsilon_{1n})$ $(\varepsilon) = (\varepsilon') = (+, +)$

We already know: $\begin{matrix} n+4 & & m+2 & & n \\ \uparrow & & \uparrow & & \uparrow \\ k_2 & & k_2 & & k_1 \end{matrix} \in \text{2Morph } E\varepsilon_{1n} \{2k_1 + 2k_2\} \rightarrow E\varepsilon_{1n}$

$$\sum_{k_1, k_2=0}^{+\infty} q^{2(k_1+k_2)} = \frac{1}{(1-q^2)^2}, \text{ BUT } \langle E^2\varepsilon_{1n}, E^2\varepsilon_{1n} \rangle = \frac{([2]!)^2}{(1-q^2)(1-q^4)} = \underbrace{(1+q^{-2})(1+q^2+q^4+q^6+\dots)}_{\substack{\text{no } q^{-4} \\ = 3 + q^2(\dots)}}$$

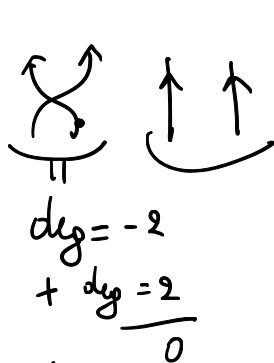
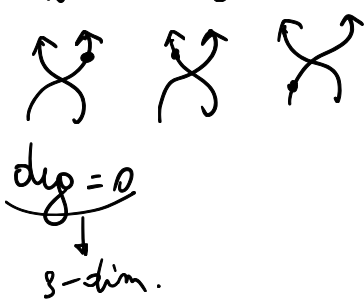
\Rightarrow We need 2-morph of $\text{deg} = -2$

\hookrightarrow = 0.

$\hookrightarrow \text{deg} = -4$

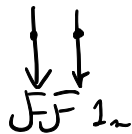
Generating set = $\left\{ \begin{matrix} \uparrow & \uparrow \\ k_2 & k_1 \end{matrix} : k_1, k_2 \in \mathbb{N} \right\} \cup \left\{ \begin{matrix} \uparrow & \uparrow \\ k_2 & k_1 \\ \parallel \\ \uparrow & \uparrow \\ k_2 & k_1 \end{matrix} : k_1, \dots, k_n \in \mathbb{N} \right\}$

Not a basis



not all lin. indep.

Similarly:



3. $\text{Hom}(F\varepsilon_{1n}, 1_n)$

$$\langle F\varepsilon_{1n}, 1_n \rangle = \langle E\varepsilon_{1n}, \underbrace{\tau(F\varepsilon_{1n})}_{\parallel} 1_n \rangle = q^{m+1} (1 + q^2 + q^4 + \dots)$$

$$\Rightarrow \mathcal{U}(F\varepsilon_{1n}(t), 1_n) = \begin{cases} q^{m+1} E & \text{1-dim if } t \in m+1 + 2\mathbb{N} \\ \{0\} & \text{otherwise} \end{cases}$$

$t = m+1$:

$t = m+3$: = 1-dim.

$\hookrightarrow \mathcal{U}(F\varepsilon_{1n}(m+1+2k)) = \langle \begin{matrix} \uparrow & \uparrow \\ k & k \end{matrix} \rangle$

$$\text{HOM}_{\mathbb{Z}}(\mathcal{E}\mathcal{F}1_n, 1_n) : \begin{array}{c} \nearrow \\ \searrow \end{array}^n \quad \text{deg} = -n+1 \leftarrow$$

$$\text{HOM}_{\mathbb{Z}}(1_n, \mathcal{F}\mathcal{E}1_n) : \begin{array}{c} \searrow \\ \nearrow \end{array}^n \quad \text{deg} = n+1$$

$$\text{HOM}_{\mathbb{Z}}(1_n, \mathcal{E}\mathcal{F}1_n) : \begin{array}{c} \nwarrow \\ \swarrow \end{array}^n \quad \text{deg} = -n+2$$

$$\begin{array}{c} \nearrow^{n+2} \\ \searrow \\ \nearrow^n \end{array} \quad \text{deg} = \text{deg}(\begin{array}{c} \nearrow^{n+2} \\ \searrow \end{array}) + \text{deg}(\begin{array}{c} \nwarrow \\ \swarrow \end{array}^n) \\ \mathcal{E}1_n(\dots) \quad = -(n+2)+1 + n+2 = 0$$

$$\begin{array}{c} \parallel \\ \mathcal{E}1_n \rightarrow \mathcal{E}1_n \end{array} \rightarrow \langle \begin{array}{c} \nearrow^{n+2} \\ \searrow \\ \nearrow^n \end{array} \rangle \\ \Rightarrow \begin{array}{c} \nearrow \\ \searrow \end{array} = \uparrow, \quad \begin{array}{c} \nwarrow \\ \swarrow \end{array} = \uparrow, \quad \begin{array}{c} \nearrow \\ \searrow \\ \nearrow \end{array} = \downarrow, \quad \begin{array}{c} \nwarrow \\ \swarrow \end{array} = \downarrow$$

\Rightarrow Bigradedness arises naturally!

4. $\text{HOM}_{\mathbb{Z}}(\mathcal{E}\mathcal{F}1_n, \mathcal{F}\mathcal{E}1_n)$

$$\langle \mathcal{E}\mathcal{F}1_n, \mathcal{F}\mathcal{E}1_n \rangle = \dots = (1+q^2)(1+q^2+q^4+q^6+\dots)^2 \\ = 1+3q^2+5q^4+\dots$$

$\bullet t=0$
We already know $\begin{array}{c} \nearrow \\ \searrow \\ \nearrow \end{array} = \begin{array}{c} \searrow \\ \nearrow \\ \searrow \end{array} = \chi_{\mathbb{Z}}^n$
 $\mathcal{E}\mathcal{F}1_n \rightarrow \mathcal{F}\mathcal{E}1_n$

Similarly: $\mathcal{F}\mathcal{E}1_n \rightarrow \mathcal{E}\mathcal{F}1_n : \chi_{\mathbb{Z}}^n$

5. $\text{HOM}_{\mathbb{Z}}(\mathcal{E}\mathcal{F}1_n, \mathcal{E}\mathcal{F}1_n)$

$$\begin{array}{c} \nearrow^{n-2} \\ \searrow \\ \nearrow^n \end{array} \quad \begin{array}{c} \nwarrow \\ \swarrow \end{array}^{l_1} \\ \begin{array}{c} \nwarrow \\ \swarrow \end{array}^{l_2} \end{array} = 2(-n+2) + 2(l_1+l_2)$$

$$\begin{array}{c} \downarrow \\ \text{deg} = 2(l_1+l_2) \end{array} \\ \Rightarrow \sum_{k_1, k_2=0}^{+\infty} q^{2(k_1+k_2)} + \sum_{l_1, l_2=0}^{+\infty} q^{2(-n+2)+2(l_1+l_2)} = \frac{1+q^{-2n+2}}{(1-q^2)^2} = \langle \mathcal{E}\mathcal{F}1_n, \mathcal{E}\mathcal{F}1_n \rangle$$

\hookrightarrow Relations:

$$\begin{array}{c} \begin{array}{c} \nwarrow \\ \swarrow \end{array} \\ \searrow \end{array} = \begin{array}{c} \uparrow \\ \downarrow \end{array} \\ \text{deg } 0 \quad \mathcal{E}\mathcal{F}1_n \rightarrow \mathcal{E}\mathcal{F}1_n$$

Theorem: Every $U(\mathbb{Z}(c) 1_n \{t\}, \mathbb{Z}(c') 1_n)$ can be built from the diagrams we've seen

- Proof:
- (Lauda 2008) Indecomposable 2-morphisms = Lusztig canonical basis elts up to grading shift
 - (Khovanov - Lauda 2010): Diagrammatic intpr. of $\langle \cdot, \cdot \rangle$: only have generating 2-morphisms

- Challenges:
- Which relations between 2-morphisms?
 - How to lift the \dot{U} -relations to nat. isomorphisms?

Finding relations between 2-morphisms:

Factor of U on cohomology rings of partial flag varieties

$\downarrow \rightarrow$ multiplication with ξ_i , $\begin{matrix} \nearrow \\ \searrow \end{matrix} \rightarrow$ divided difference operator ∂_i

NHA = nilHecke algebra = $\langle \xi_i, \partial_j : i = 1, \dots, a, j = 1, \dots, a-1 \rangle$

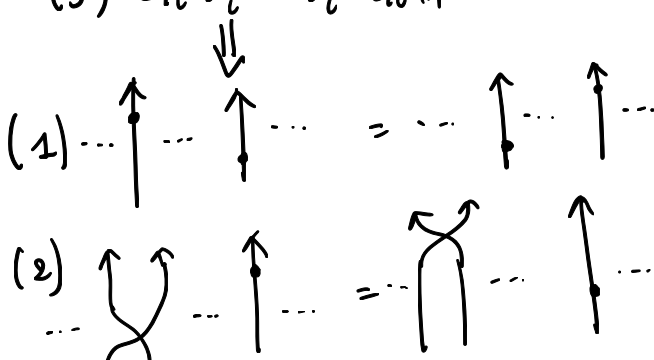
(1) $[\xi_i, \xi_j] = 0$

(2) $[\partial_i, \xi_j] = [\partial_i, \partial_j] = 0$ if $|i-j| > 1$

(3) $\partial_i^2 = 0$

(4) $\partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1}$

(5) $\xi_i \partial_i - \partial_i \xi_{i+1} = \partial_i \xi_i - \xi_{i+1} \partial_i = 1$



(3) $\begin{matrix} \nearrow \\ \searrow \end{matrix} \begin{matrix} \nearrow \\ \searrow \end{matrix} = 0$

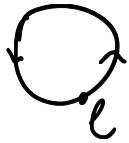
(4) $\begin{matrix} \nearrow \\ \searrow \end{matrix} \begin{matrix} \nearrow \\ \searrow \end{matrix} = \begin{matrix} \nearrow \\ \searrow \end{matrix} \begin{matrix} \nearrow \\ \searrow \end{matrix}$

(5) $\begin{matrix} \nearrow \\ \searrow \end{matrix} - \begin{matrix} \searrow \\ \nearrow \end{matrix} = \begin{matrix} \nearrow \\ \searrow \end{matrix} - \begin{matrix} \searrow \\ \nearrow \end{matrix} = \uparrow \uparrow$

\Downarrow repeatedly

$k \begin{matrix} \nearrow \\ \searrow \end{matrix} - \begin{matrix} \searrow \\ \nearrow \end{matrix} k = k \begin{matrix} \nearrow \\ \searrow \end{matrix} - \begin{matrix} \searrow \\ \nearrow \end{matrix} k = \sum_{\substack{l_1, l_2 \in \mathbb{N} \\ l_1 + l_2 = k-1}} \begin{matrix} \nearrow \\ \searrow \end{matrix} \uparrow \uparrow \begin{matrix} \searrow \\ \nearrow \end{matrix} l_2$

Bubbles:



$$dy = 2(n+1) + 2l$$

$$\in \text{HOM}_i(\mathbb{1}_n, \mathbb{1}_n)$$

$$dy = 2(-n+1) + 2k$$

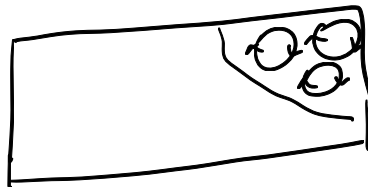
$$\text{BUT } \langle \mathbb{1}_m, \mathbb{1}_n \rangle = 1 = \sum_{t \in \mathbb{Z}} q^t \dim(\text{HOM}_i(\mathbb{1}_m(t), \mathbb{1}_n))$$

\Rightarrow Suggests that bubbles of $dy \neq 0$ should = 0. \searrow

BUT categorical action: action $\neq 0$

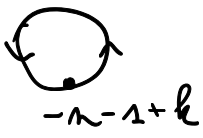
Prop.: Any diagram can be reduced to lin. comb. of diagrams where all bubbles:

- non-nested
- same orientation
- far right corner



\Rightarrow Graded dimensions can be renormalized to account for contribution of bubbles

$$\hookrightarrow \langle E_{(k)} \mathbb{1}_m, E_{(l)} \mathbb{1}_n \rangle = \frac{\text{gdim}(\text{HOM}_i(E_{(k)} \mathbb{1}_m, E_{(l)} \mathbb{1}_n))}{\text{gdim}(\text{HOM}_i(\mathbb{1}_m, \mathbb{1}_n))}$$

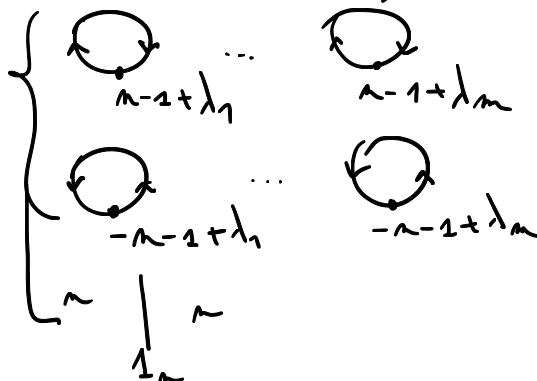


$$= \begin{cases} 0 & \text{if } k \leq 0 \text{ ("fake bubbles")} \\ \text{multiple of } \mathbb{1}_{\mathbb{1}_n} & \text{if } k = 0. \end{cases}$$

$$\Downarrow \text{deg} = 2k$$

Given λ partition = $(\lambda_1, \lambda_2, \dots, \lambda_m)$

Def.: $e_{\lambda, n} :=$



if $n > 0 \wedge \lambda \neq \emptyset$

if $n \leq 0 \wedge \lambda \neq \emptyset$

if $n = 0 \wedge \lambda = \emptyset$

Lifting the \bar{U} -relations to isomorphisms

Goal: $\mathcal{E}F_{1_n} \cong \mathcal{F}\mathcal{E}_{1_n} \oplus 1_n\{n-2\} \oplus 1_n\{n-3\} \oplus \dots \oplus 1_n\{-n+1\}$ if $n \geq 0$

$\mathcal{F}\mathcal{E}_{1_n} \cong \mathcal{E}F_{1_n} \oplus 1_n\{-n-1\} \oplus 1_n\{-n-3\} \oplus \dots \oplus 1_n\{n+1\}$ if $n < 0$

Assume $n \geq 0$. We already know 2-morphism $\mathcal{F}\mathcal{E}_{1_n} \oplus 1_n\{n-2\} \oplus \dots \rightarrow \mathcal{E}F_{1_n}$

$\zeta_+ := \chi_n \oplus \begin{matrix} \curvearrowright \\ n-2 \end{matrix} \oplus \begin{matrix} \curvearrowright \\ n-2 \end{matrix} \oplus \dots \oplus \begin{matrix} \curvearrowright \\ \end{matrix} \oplus \begin{matrix} \curvearrowright \\ \end{matrix}$

\downarrow
 $\text{deg} = 2(n-2) + (-n+1)$
 $= n-1$
 $1_n\{n-1\} \rightarrow \mathcal{E}F_{1_n}$

If an inverse exists, then it must be:

$\bar{\zeta}_+ := \beta_n \chi_n \oplus \bigoplus_{l=0}^{n-1} \bigoplus_{\substack{\lambda \text{ partition} \\ |\lambda|+j=l}} \alpha_{\lambda}^l(n) e_{\lambda, n} \begin{matrix} \curvearrowright \\ \end{matrix}$

for certain $\beta_n, \alpha_{\lambda}^l(n)$ scalars

Similarly for $n < 0$:

$\zeta_- := \chi_n \oplus \begin{matrix} \curvearrowleft \\ -n-1 \end{matrix} \oplus \dots \oplus \begin{matrix} \curvearrowleft \\ \end{matrix}$

$\bar{\zeta}_- = \beta_n \chi_n \oplus \bigoplus_{l=0}^{-n-1} \bigoplus_{\substack{\lambda \text{ partition} \\ |\lambda|+j=l}} \alpha_{\lambda}^l(n) e_{\lambda, n} \begin{matrix} \curvearrowleft \\ \end{matrix}$

Theorem: $\bar{\zeta}_+$ and $\bar{\zeta}_-$ are inverse of ζ_+, ζ_- resp., and hence

ζ_+, ζ_- are sought isomorphisms

\Leftrightarrow the 2-morphisms satisfy the relations of Tables A, B.

Relations for $n \geq 0$	
(A1)	$\delta_{b,0} = \sum_{\lambda: \lambda \leq b} \alpha_{\lambda}^{\ell}(n) e_{\lambda,n} \text{ (diagram: a circle with a dot at the bottom, labeled } n \text{ at the top and } n-1+b- \lambda \text{ at the bottom)}$
(A2)	$n \text{ (diagram: two vertical lines with arrows pointing down)} = \beta_n \text{ (diagram: two vertical lines with arrows pointing up, crossing each other)}$
(A3)	$\beta_n \text{ (diagram: a figure-eight loop with a dot at the bottom, labeled } n-1-\ell \text{ at the bottom)} = 0$
(A4)	$\sum_{\lambda} \alpha_{\lambda}^{\ell}(n) e_{\lambda,n} \text{ (diagram: a figure-eight loop with a dot at the top, labeled } n-1-\ell \text{ at the top)} = 0$
(A5)	$n \text{ (diagram: two vertical lines with arrows pointing up)} = \beta_n \text{ (diagram: two vertical lines with arrows pointing down, crossing each other)} + \sum_{\substack{f_1+f_2+ \lambda \\ =n-1}} \alpha_{\lambda}^{ \lambda +f_2}(n) e_{\lambda,n} \text{ (diagram: two arcs, one above the other, labeled } n \text{ at the top and } f_1, f_2 \text{ at the ends)}$

Note that relations **A1**, **A3**, and **A4** are only valid for $n > 0$.

Relations for $n \geq 0$	
(B1)	$\delta_{b,0} = \sum_{\lambda: \lambda \leq b} \alpha_{\lambda}^{\ell}(n) e_{\lambda,n} \text{ (diagram: a circle with a dot at the bottom, labeled } n \text{ at the top and } -n-1+b- \lambda \text{ at the bottom)}$
(B2)	$n \text{ (diagram: two vertical lines with arrows pointing down)} = \beta_n \text{ (diagram: two vertical lines crossing twice, labeled } n \text{ at the top)}$
(B3)	$\beta_n \text{ (diagram: a figure-eight loop with a dot at the bottom, labeled } n \text{ at the top and } -n-1-\ell \text{ at the bottom)} = 0$
(B4)	$\sum_{\lambda} \alpha_{\lambda}^{\ell}(n) e_{\lambda,n} \text{ (diagram: a figure-eight loop with a dot at the top, labeled } -n-1-\ell \text{ at the top and } n \text{ at the bottom)} = 0$
(B5)	$n \text{ (diagram: two vertical lines with arrows pointing down)} = \beta_n \text{ (diagram: two vertical lines crossing twice, labeled } n \text{ at the top)} + \sum_{\substack{g_1+g_2+ \lambda \\ =-n-1}} \alpha_{\lambda}^{ \lambda +g_2}(n) e_{\lambda,n} \text{ (diagram: two arcs with dots, labeled } n \text{ at the top and } g_1, g_2 \text{ at the dots)}$

Note that relations **B1**, **B3**, and **B4** are only valid for $n < 0$.

$$u_{\chi} \stackrel{!}{=} \tilde{u}$$