Categorification of quantum sl_2: Part II

Recall from last week:

- $\dot{U}$ = idempotent part of $U_q(sl_2)$ = associative non-unital algebra over $\mathbb{Q}(q)$ with generators 1_n, $1_{n+2}E1_n$, $1_{n-2}F1_n$ $(n \in \mathbb{Z})$

  and relations: $\overset{\shortparallel}{E1_n} = 1_{n+2}E$ $\qquad \overset{\shortparallel}{F1_n} = 1_{n-2}F$

  $\rightarrow$ • $1_n 1_m = \delta_{nm} 1_n$ $\qquad \rightarrow 1_m E 1_n = 0$ if $m \neq n+2$

  $\rightarrow$ • $[E,F]1_n = [n]1_n = \dfrac{q^n - q^{-n}}{q-q^{-1}} \cdot 1_n$

  $\hookrightarrow = (q^{n-1} + q^{n-3} + \dots + q^{-n+1})1_n$

- 2-Category $\mathcal{U}$ with:
  - objects: $n \in \mathbb{Z}$
  - 1-Morphisms (n,m):

    $\rightarrow 1_m \mathcal{E}_{(\varepsilon_1)} 1_n \{t_1\} \oplus \dots \oplus 1_m \mathcal{E}_{(\varepsilon_N)} 1_n \{t_N\}$

    $\hookrightarrow$ generating 1-morphisms: 1_n, $\mathcal{E}$, $\mathcal{F}$

    $\hookrightarrow$ $t_1, \dots, t_N \in \mathbb{Z}$

    $\hookrightarrow$ $\varepsilon = (\varepsilon_1, \dots, \varepsilon_m)$, $\varepsilon_i \in \{+,-\}$, $\mathcal{E}_{(\varepsilon)} = \mathcal{E}_{\varepsilon_1} \cdots \mathcal{E}_{\varepsilon_m}$,

    $\mathcal{E}_+ = \mathcal{E}$, $\mathcal{E}_- = \mathcal{F}$, $m = n + 2\sum_{i=1}^{m} \varepsilon_i \cdot 1$
  - 2-Morphisms

$\hookrightarrow$ $K_0(1\mathrm{Morph}_{\mathcal{U}}(n,m)) = 1_m \dot{U} 1_n$, $\mathbb{Z}[q,q^{-1}]$-module

$\hookrightarrow$ $[f\{t\}] = q^t [f]$

$\hookrightarrow$ $\mathcal{U}$ = additive 2-category

$\hookrightarrow$ Composition of 1-morphisms in $\mathcal{U}$ $\Leftrightarrow$ multiplication in $\dot{U}$

$\mathbb{Z}[q,q^{-1}]$-semilinear form on $\dot{U}$

$[a]! = [a][a-1]\dots[1]$

- $\langle E^a 1_n, E^a 1_n \rangle = \langle F^a 1_n, F^a 1_n \rangle = \dfrac{([a]!)^2}{\prod_{j=1}^{a} 1-q^{2j}}$, $a \in \mathbb{N}$

- $\langle ux, y \rangle = \langle x, \tau(u) y \rangle$, $u, x, y \in \dot{U}$

  $\rightarrow$ with $\tau(E1_n) = q^{-n+1} 1_n F$, $\tau(F1_n) = q^{n+1} 1_n E$

Write $\mathcal{U}(f,g) = 2\mathrm{Morph}_{\mathcal{U}}(f,g)$, $\rightarrow \mathrm{HOM}_{\mathcal{U}}(f,g) = \bigoplus_{t \in \mathbb{Z}} \mathcal{U}(f\{t\}, g)$ $\qquad f,g \in 1\mathrm{Morph}(m,n)$

$\langle [f], [g] \rangle = \mathrm{gdim}(\mathrm{HOM}_{\mathcal{U}}(f,g)) = \sum_{t \in \mathbb{Z}} q^t \dim(\mathcal{U}(f\{t\}, g))$ $\Leftarrow$

Find the vector space $\mathcal{U}(\mathcal{E}_{(\varepsilon)} 1_n\{t\}, \mathcal{E}_{(\varepsilon')} 1_n\{t'\})$, $\forall \varepsilon, \varepsilon', n, t', t$

W.l.o.g.: $t'=0$

$$\mathrm{gdim}(\mathrm{HOM}_{\mathcal{U}}(\mathcal{E}_{(\varepsilon)}1_n\{t\}, \mathcal{E}_{(\varepsilon')}1_n\{t'\}))$$

$$= \mathrm{gdim}(\mathrm{HOM}_{\mathcal{U}}(\mathcal{E}_{(\varepsilon)}1_n\{t-t'\}, \mathcal{E}_{(\varepsilon')}1_n))$$

Ansatz: $\langle E_{(\varepsilon)}1_n, E_{(\varepsilon')}1_n\rangle = \mathrm{gdim}(\mathrm{HOM}_{\mathcal U}(\mathcal E_{(\varepsilon)}1_n, \mathcal E_{(\varepsilon')}1_n))$

$$= \sum_{t\in\mathbb Z} q^t \dim(\mathcal U(\mathcal E_{(\varepsilon)}1_n\{t\}, \mathcal E_{(\varepsilon')}1_n))$$

1. $\mathrm{HOM}_{\mathcal U}(\mathcal E 1_n, \mathcal E 1_n)$, $(\varepsilon) = (\varepsilon') = (+)$

$$\sum_{t\in\mathbb Z} q^t \dim(\mathcal U(\mathcal E 1_n\{t\}, \mathcal E 1_n)) = \langle E1_n, E1_n\rangle = \frac{1}{1-q^2} = 1+q^2+q^4+q^6+\dots$$

$$\Rightarrow \forall t\in\mathbb Z:\ \dim(\mathcal U(\mathcal E 1_n\{t\}, \mathcal E 1_n)) = \begin{cases} 0 & \text{if } t<0 \text{ or } t \text{ is odd} \\ 1 & \text{if } t\in 2\mathbb N \end{cases}$$

$t=0$: $\mathcal U(\mathcal E 1_n, \mathcal E 1_n) =$ k-linear span of $1_{\mathcal E 1_n}$ $\hookrightarrow$ $n+2 \uparrow n$

$t=2$: 2-morph $\mathcal E 1_n\{2\} \to \mathcal E 1_n$ $\quad n+2 \ \overset{\uparrow}{\bullet}\ n$

$\deg\left(n+2 \ \overset{\uparrow}{\bullet}\ n\right) = 2$

$t=4$: $\mathcal E 1_n\{4\} \to \mathcal E 1_n$: Vertical composition $\quad \overset{\mathcal E 1_n}{\underset{\mathcal E 1_n\{2\}}{n+2 \ \overset{\uparrow}{\bullet}\ n}}$, $\overset{\mathcal E 1_n\{2\}}{\underset{\mathcal E 1_n\{4\}}{n+2 \ \overset{\uparrow}{\bullet}\ n}}$

$$\deg\left(n+2 \ \overset{\uparrow}{\underset{\bullet}{\bullet}}\ n\right) = 4$$

$$n+2 \ \overset{\uparrow}{\bullet}_k\ n = \overset{\uparrow}{\left.\begin{matrix}\bullet\\ \bullet\\ \vdots\end{matrix}\right\}} k \text{ dots}$$

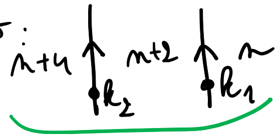
$$\Rightarrow \mathcal U(\mathcal E 1_n\{2k\}, \mathcal E 1_n) = k\text{-lin. span of } n+2 \ \overset{\uparrow}{\bullet}_k\ n$$

$k\in\mathbb N\setminus\{0\}$

Similarly for $\mathcal F 1_n, \mathcal F 1_n$: $\quad \mathcal U(\mathcal F 1_n, \mathcal F 1_n) = \langle n-2 \downarrow n \rangle$

$\mathcal U(\mathcal F 1_n\{2k\}, \mathcal F 1_n) = \langle n-2 \ \overset{\downarrow}{\bullet}_k\ n \rangle$

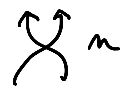
2. $\mathrm{HOM}_{\mathcal{U}}(\mathcal{E}\mathcal{E}1_n, \mathcal{E}\mathcal{E}1_n)$ $\quad (\varepsilon) = (\varepsilon') = (+,+)$

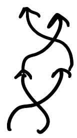
We already know:  $\in$ 2Morph $\mathcal{E}\mathcal{E}1_n\{2k_1+2k_2\} \to \mathcal{E}\mathcal{E}1_n$

$$\sum_{k_1,k_2=0}^{+\infty} q^{2(k_1+k_2)} = \frac{1}{(1-q^2)^2}, \text{ BUT } \langle E^2 1_n, E^2 1_n \rangle = \frac{([2]!)^2}{(1-q^2)(1-q^4)}$$

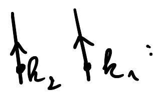
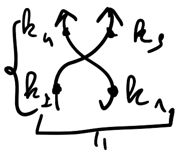
$$= \underbrace{(1+q^{-2})(1+q^2+q^4+q^6+\dots)^2}_{\text{no } q^{-4}}$$

$$= 3 + q^2(\dots)$$

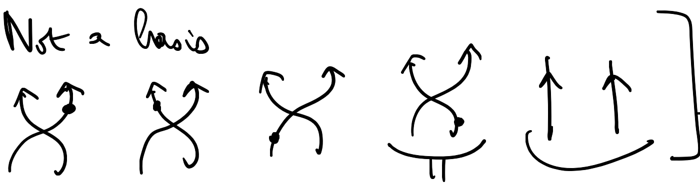
$\Rightarrow$ We need 2-morph of deg $= -2$ 

↳  $= 0.$

↳ deg $= -4$

Generating set $= \{$  $: k_1, k_2 \in \mathbb{N}\} \cup \{$  $: k_1, \dots, k_4 \in \mathbb{N}\}$

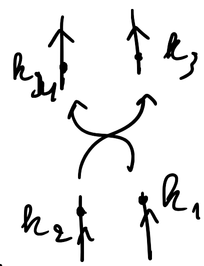
Not a basis

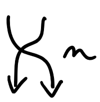
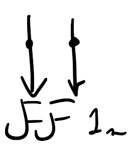


deg $= 0$ → 3-dim.

deg $= -2$ + deg $= 2$ → 0

→ not all lin. indep.

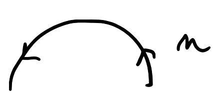


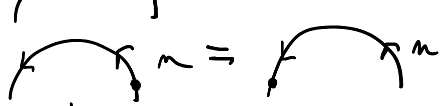
Similarly:  

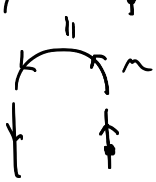
3. $\mathrm{HOM}_{\mathcal{U}}(\mathcal{F}\mathcal{E}1_n, 1_n)$

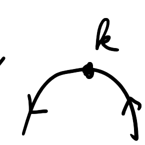
$$\langle FE1_n, 1_n \rangle = \langle E1_n, \underbrace{\tau(F1_n)}_{= q^{n+1}E} 1_n \rangle = q^{n+1}(1+q^2+q^4+\dots)$$

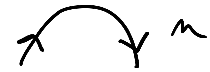
$$\Rightarrow \mathcal{U}(\mathcal{F}\mathcal{E}1_n\{t\}, 1_n) = \begin{cases} 1\text{-dim} & \text{if } t \in n+1+2\mathbb{N} \\ \{0\} & \text{otherwise} \end{cases}$$

$t = n+1$: 

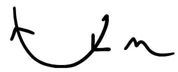
$t = n+3$:  1-dim.

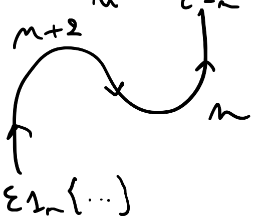


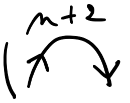
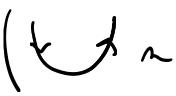
↳ $\mathcal{U}(\mathcal{F}\mathcal{E}1_n\{n+1+2k\}) = \langle$  $\rangle$

$\mathrm{HOM}(\mathcal{E}\mathcal{F}1_n, 1_n)$ $\ni$:  $\deg = -n+1$ $\leftarrow$

$\mathrm{HOM}(1_n, \mathcal{F}\mathcal{E}1_n)$ $\ni$:  $\deg = n+1$

$\mathrm{HOM}(1_n, \mathcal{E}\mathcal{F}1_n)$ $\ni$:  $\deg = -n+1$

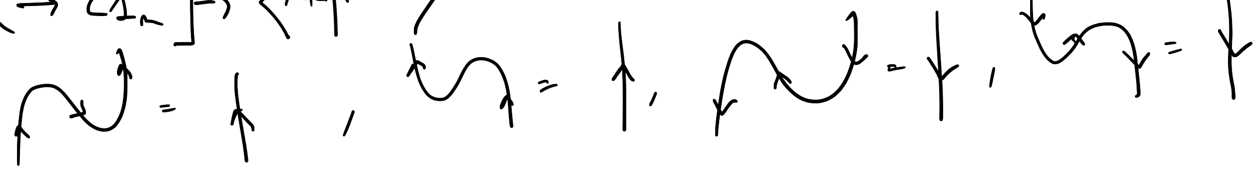


$\deg = \deg($  $) + \deg($  $)$

$= -(n+2)+1+n+1 = 0$

$\shortparallel$

$[\mathcal{E}1_n \to \mathcal{E}1_n] \to$ 

$\Rightarrow$ 

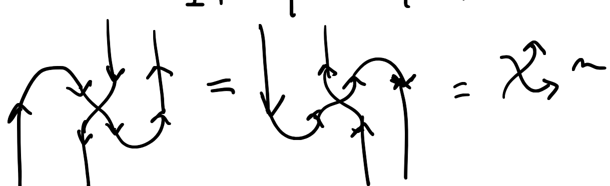
$\Rightarrow$ Biadjointness arises naturally!

4. $\mathrm{HOM}(\mathcal{E}\mathcal{F}1_n, \mathcal{F}\mathcal{E}1_n)$ $\ni$

$\langle EF1_n, FE1_n \rangle = \dots = (1+q^2)(1+q^2+q^4+q^6+\dots)^2$

$= 1 + 3q^2 + 5q^4 + \dots$

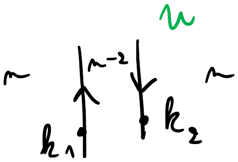
- $t=0$

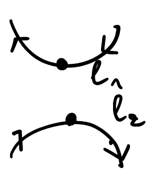
We already know 

$\mathcal{E}\mathcal{F}1_n \to \mathcal{F}\mathcal{E}1_n$

Similarly: $\mathcal{F}\mathcal{E}1_n \to \mathcal{E}\mathcal{F}1_n$: 

5. $\mathrm{HOM}(\mathcal{E}\mathcal{F}1_n, \mathcal{E}\mathcal{F}1_n)$: $\ni$



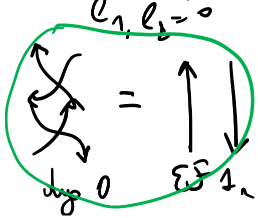


$= 2(-n+1) + 2(\ell_1+\ell_2)$

$\Downarrow$

$\deg = 2(k_1+k_2)$

$$\Rightarrow \sum_{k_1,k_2=0}^{+\infty} q^{2(k_1+k_2)} + \sum_{\ell_1,\ell_2=0}^{+\infty} q^{2(-n+1)+2(\ell_1+\ell_2)} = \frac{1+q^{-2n+2}}{(1-q^2)^2} \overset{!}{=} \langle EF1_n, EF1_n \rangle$$

$\hookrightarrow$ Relations: 

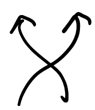
Theorem: Every $\mathcal{U}(\mathcal{E}_{(\varepsilon)} 1_n\{t\}, \mathcal{E}_{(\varepsilon')} 1_n)$ can be built from the diagrams we've seen

Proof:
- (Lauda 2008) Indecomposable 1-morphisms = Lusztig canonical basis elts up to grading shift
- (Khovanov – Lauda 2010): Diagrammatic interpr. of $\langle \cdot , \cdot \rangle$: only have generating 2-morphisms

Challenges:
- Which relations between 2-morphisms?
- How to lift the $\dot{U}$-relations to nat. isomorphisms?

Finding relations between 2-morphisms:

$\exists$ action of $\mathcal{U}$ on cohomology rings of partial flag varieties

$\hookrightarrow$  $\rightsquigarrow$ multiplication with x_i,  $\rightsquigarrow$ divided difference operator ∂_i

NH_a = nilHecke algebra = $\langle x_i, \partial_j : i = 1, \dots, a, \ j = 1, \dots, a-1 \rangle$

(1) $[x_i, x_j] = 0$

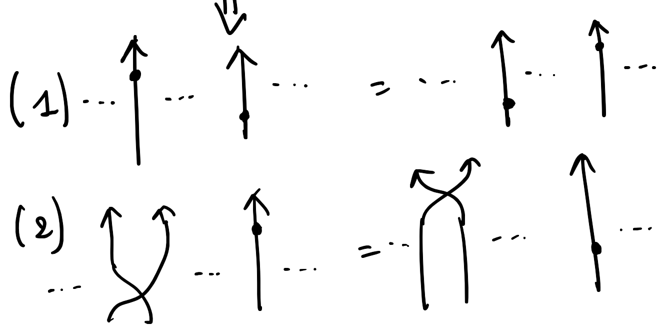
(2) $[\partial_i, x_j] = [\partial_i, \partial_j] = 0$ if $|i-j| > 1$

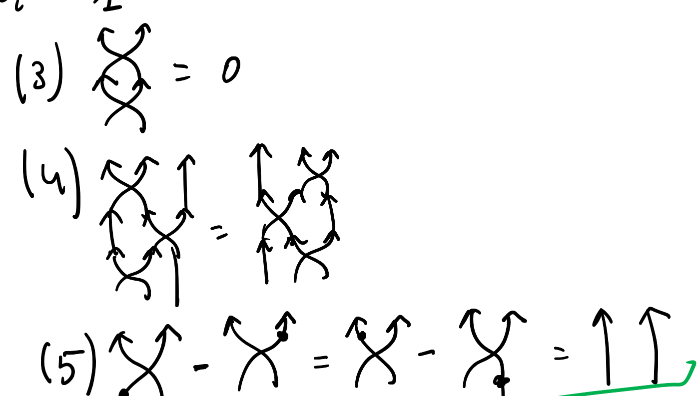
(3) $\partial_i^2 = 0$

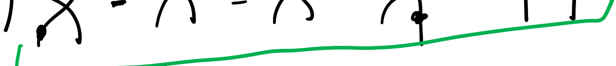
(4) $\partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1}$

(5) $x_i \partial_i - \partial_i x_{i+1} = \partial_i x_i - x_{i+1} \partial_i = 1$

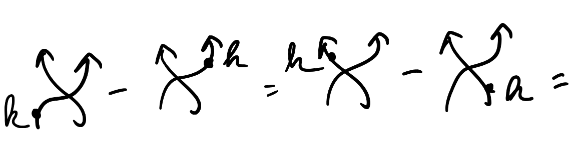
$\Downarrow$

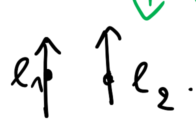




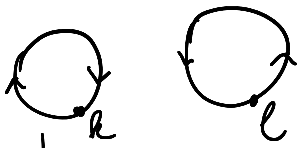


$\Downarrow$ repeating



$= \sum_{\substack{l_1, l_2 \in \mathbb{N} \\ l_1 + l_2 = k-1}}$ 

Bubbles:



$\deg = 2(n+1)+2\ell$

$\in \mathrm{HOM}_{\mathcal{U}}(1_n, 1_n)$

$\deg = 2(-n+1)+2k$

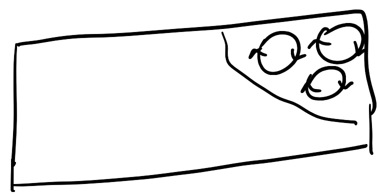
BUT $\langle 1_n, 1_n\rangle = 1 = \sum_{t\in\mathbb{Z}} q^t \dim\left(\mathcal{U}(1_n\{t\}, 1_n)\right)$

$\Rightarrow$ Suggests that bubbles of $\deg \neq 0$ should $= 0$. $\lightning$

BUT categorical action: action $\neq 0$

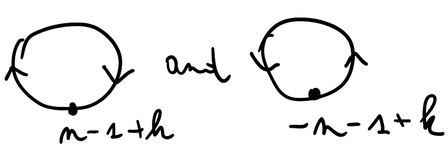
Prop.: Any diagram can be reduced to lin. comb. of diagrams where all bubbles:

- non-nested
- same orientation
- far right corner



$\Rightarrow$ Graded dimensions can be renormalized to account for contribution of bubbles

$\hookrightarrow$ $$\langle E_{(\varepsilon)}1_n, E_{(\varepsilon')}1_n\rangle = \frac{\mathrm{gdim}\left(\mathrm{HOM}_{\mathcal{U}}(\mathcal{E}_{(\varepsilon)}1_n, \mathcal{E}_{(\varepsilon')}1_n)\right)}{\mathrm{gdim}\left(\mathrm{HOM}_{\mathcal{U}}(1_n, 1_n)\right)}$$

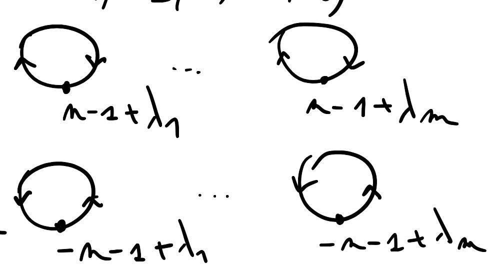


$$= \begin{cases} 0 & \text{if } k<0 \text{ ("fake bubbles")} \\ \text{multiple of } 1_{1_n} & \text{if } k=0. \end{cases}$$

$\deg = 2k$

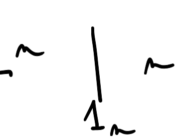
Given λ partition $= (\lambda_1, \lambda_2, \dots, \lambda_m)$

Def.: $e_{\lambda,n} :=$



if $n>0 \wedge \lambda\neq\emptyset$

if $n<0 \wedge \lambda\neq\emptyset$



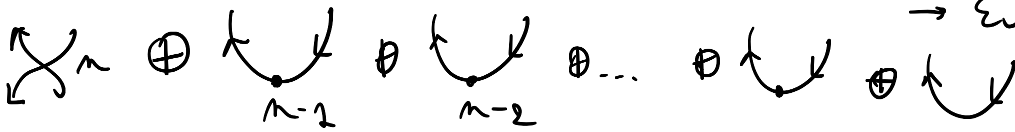
if $n=0 \vee \lambda=\emptyset$

Lifting the $\dot{U}$-relations to isomorphisms

Goal: $\mathcal{E}\mathcal{F}1_n \cong \mathcal{F}\mathcal{E}1_n \oplus 1_n\{n-1\} \oplus 1_n\{n-3\} \oplus \dots \oplus 1_n\{-n+1\}$ if $n \geq 0$

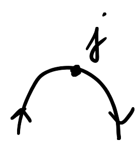
$\mathcal{F}\mathcal{E}1_n \cong \mathcal{E}\mathcal{F}1_n \oplus 1_n\{-n-1\} \oplus 1_n\{-n-3\} \oplus \dots \oplus 1_n\{n+1\}$ if $n < 0$

Assume $n \geq 0$. We already know 2-morphism $\mathcal{F}\mathcal{E}1_n \oplus 1_n\{n-1\} \oplus \dots \to \mathcal{E}\mathcal{F}1_n$

$\zeta_+ :=$ 

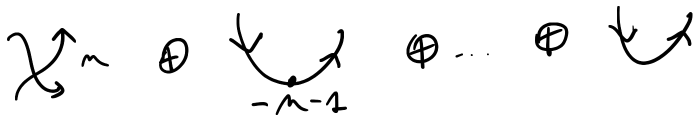
$\deg = 2(n-1) + (-n+1)$
$= n-1$
$1_n\{n-1\} \to \mathcal{E}\mathcal{F}1_n$

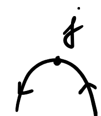
If an inverse exists, then it must be:

$\overline{\zeta_+} := \beta_n$  $\oplus \bigoplus_{l=0}^{n-1} \bigoplus_{\substack{\lambda \text{ partition}, j \in \mathbb{N} \\ |\lambda| + j = l}} \alpha_\lambda^l(n)\, e_{\lambda,n}$ 

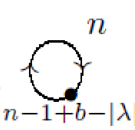
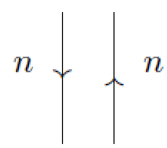
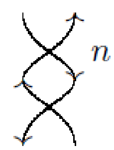
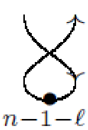
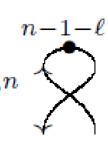
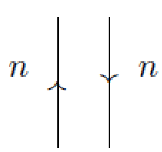
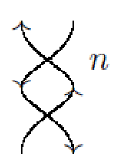
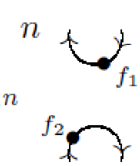
for certain β_n, $\alpha_\lambda^l(n)$ scalars

Similarly for $n < 0$:

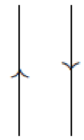
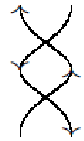
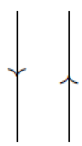
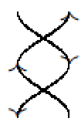
$\zeta_- :=$ 

$\overline{\zeta_-} = \beta_n$  $\oplus \bigoplus_{l=0}^{-n-1} \bigoplus_{\substack{\lambda \text{ partition}, j \in \mathbb{N} \\ |\lambda| + j = l}} \alpha_\lambda^l(n)\, e_{\lambda,n}$ 

Theorem: $\overline{\zeta_+}$ and $\overline{\zeta_-}$ are inverse of ζ_+, ζ_- resp., and hence ζ_+, ζ_- are sought isomorphisms $\iff$ the 2-morphisms satisfy the relations of Tables A, B.

Relations for $n \geq 0$	
(A1)	$\delta_{b,0} = \sum_{\lambda:\lvert\lambda\rvert\leq b} \alpha^{\ell}_{\lambda}(n)\, e_{\lambda,n}$ 
(A2)	 $=\ \beta_n$ 
(A3)	β_n  $n \ = \ 0$
(A4)	$\sum_{\lambda} \alpha^{\ell}_{\lambda}(n)\, e_{\lambda,n}$  $n \ = \ 0$
(A5)	 $=\ \beta_n$  $+ \sum_{\substack{f_1+f_2+\lvert\lambda\rvert \\ =n-1}} \alpha^{\lvert\lambda\rvert+f_2}_{\lambda}(n) e_{\lambda,n}$ 

Note that relations **A1**, **A3**, and **A4** are only valid for $n > 0$.

Relations for $n \geq 0$	
(B1)	$\delta_{b,0} = \sum_{\lambda:\lvert\lambda\rvert \leq b} \alpha^{\ell}_{\lambda}(n)\; e_{\lambda,n}$  n, $-n-1+b-\lvert\lambda\rvert$
(B2)	n  n $=$ β_n  n
(B3)	β_n  $-n-1-\ell$ n $=$ 0
(B4)	$\sum_{\lambda} \alpha^{\ell}_{\lambda}(n)\; e_{\lambda,n}$  $-n-1-\ell$ n $=$ 0
(B5)	n  n $=$ β_n  n $+$ $\sum_{\substack{g_1+g_2+\lvert\lambda\rvert \\ =-n-1}} \alpha^{\lvert\lambda\rvert+g_2}_{\lambda}(n)\; e_{\lambda,n}$ n  g_1 g_2

Note that relations **B1**, **B3**, and **B4** are only valid for $n < 0$.

$\mathcal{U}_\chi \cong \dot{\mathcal{U}}$