Categorification of $U_q(sl_2)$: Part 1

Def. $U_q(sl_2)$: associative $\mathbb{Q}(q)$-algebra, unital

Generators: E, F, K, K^{-1}

Relations: $KK^{-1} = K^{-1}K = 1$

$KE = q^2 EK, \quad KF = q^{-2} FK$

$$[E,F] = \frac{K-K^{-1}}{q-q^{-1}}$$

$\rightsquigarrow sl_2 : e, f, h : [h,e) = 2e \rightsquigarrow K = q^h$

Repr. theory: • Let V fin-dim. $U_q(sl_2)$-mod

$V[n] := \{v \in V : K.v = q^n v\}$ weight space

$n \in \mathbb{Z}$

$v \in V[n)$: $K.(E.v) = q^2 E(\underbrace{K.v}_{q^n v}) = q^{n+2} \underbrace{E.v}_{\hookrightarrow \in V[n+2]}$, $K.(F.v) = q^{n-2} \underbrace{F.v}_{\hookrightarrow \in V[n-2]}$

$$\begin{tikzcd}
\cdots \arrow[r, bend left, "E"] & V[n-2] \arrow[l, bend left, "F"] \arrow[r, bend left, "E"] & V[n] \arrow[l, bend left, "F"] \arrow[r, bend left, "E"] & V[n+2] \arrow[l, bend left, "F"] \arrow[r, bend left, "E"] & \cdots \arrow[l, bend left, "F"]
\end{tikzcd}$$

• Irred. $U_q(sl_2)$-repr. of dim. $j+1$: V^j

$\hookrightarrow j \in \mathbb{N}$

$\hookrightarrow$ Highest weight vector v_0^j, weight $= j$

$\underbrace{v_k^j} := \dfrac{F^k . v_0^j}{\underbrace{[k]!}} \rightarrow \{v_k^j : k \in \{0, \ldots, j\}\}$ basis for V^j

$\hookrightarrow [k] = \dfrac{q^k - q^{-k}}{q - q^{-1}} \xrightarrow{q \to 1} k, \quad [k]! = [k][k-1] \ldots [1]$

$E . v_k^j = [j-k+1] . v_{k-1}^j \qquad (v_{-1}^j := 0)$

$F . v_k^j = [k+1] \, v_{k+1}^j$

All weight spaces = 1-dim.

$h\cancel{K} . v_k^j = (j-2k) v_k^j$

$K . v_j^k = q^{j-2k} v_k^j$

Lusztig's idempotent form of $U_q(sl_2)$: $\underline{\dot{U}}$

$1 \longrightarrow 1_n$ mutually orthog. idempotents

$n \in \mathbb{Z}$ ↳ $\forall\, V \in U_q(sl_2)$-mod: $\pi_V(1_n)$

$= $ projection onto $\underline{V[n]}$

↳ $1_n 1_m = \delta_{n,m} 1_n$

→ $K.\,1_n = q^n 1_n$

$E.\,1_n = 1_{n+2}E = 1_{n+2}E 1_n$

$F.\,1_n = 1_{n-2}F = 1_{n-2}F1_n$

$$[E,F].\,1_n = \frac{q^n - q^{-n}}{q - q^{-1}}\, 1_n = [n]\, 1_n$$

$\Rightarrow E_{(\varepsilon)} := E_{\varepsilon_1} E_{\varepsilon_2} \cdots E_{\varepsilon_M} \Rightarrow \{\underline{1_m E_{(\varepsilon)} 1_n} : n, m \in \mathbb{Z} : \varepsilon\}$ generates for $\dot{U}$

$\varepsilon = (\varepsilon_1, \dots, \varepsilon_M)$, $\varepsilon_i \in \{+,-\}$, $E_+ = E$, $E_- = F$

$$1_n E_{(\varepsilon)} 1_m \neq 0 \Rightarrow n = m + 2\sum_{i=1}^{M} \varepsilon_i \cdot 1$$

Cat. of $\dot{U}$-modules = equiv. to cat. of $U_q(sl_2)$-mod with weight decomp.

↳ $V = \bigoplus_{n \in \mathbb{Z}} V[n]$.

$\dot{U}$ = category

Objects: $n \in \mathbb{Z}$

→ Morphisms: $\mathrm{Hom}_{\dot{U}}(n,m) = 1_m \dot{U} 1_n$ ↗ $1_m E_{(\varepsilon)} 1_n$

↳ identity: 1_n

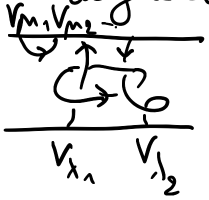
↳ composition: $1_m E_{(\varepsilon)} \underbrace{1_n} \circ \underbrace{1_{m'}} E_{(\varepsilon')} 1_{n'} = \delta_{n,m'} 1_m E_{(\varepsilon)} E_{(\varepsilon')} 1_{n'}$

⇝ Categorification of $\dot{U}$ = $\underline{\text{2-category}}$.

Motivation:

- Reshetikhin-Turaev invariants

tangle diagram T



$f(T): \overbrace{V_{\lambda_1} \otimes V_{\lambda_2}} \to \overline{V_{\mu_1} \otimes \dots \otimes V_{\mu_4}}$

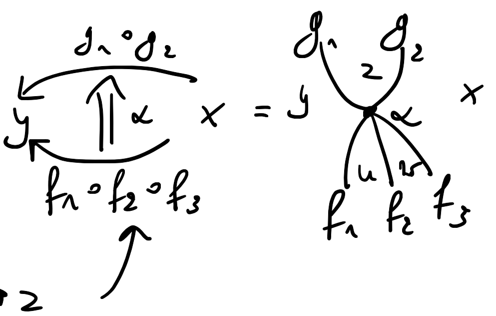
↳ $U_q(\mathfrak{g})$-intertwiner

↳ Jones pol.

- Higher structure

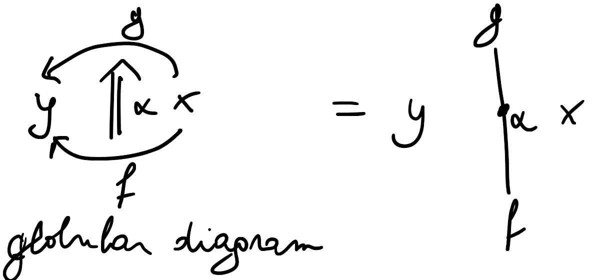
2- categories:

- Objects: $x, y, z, \dots$
- 1- Morphisms: $f : x \to y$
  - ↳ 1_x
  - ↳ $f : x \to y,\ g : y \to z :\ g \circ f : x \to z$



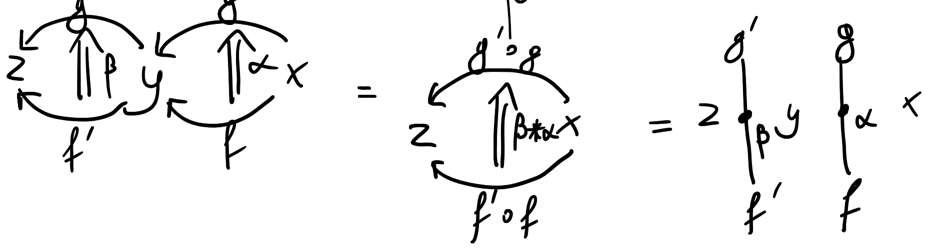
- 2- morphisms: $\alpha : f \Rightarrow g$

$f : x \to y,\ g : x \to y$

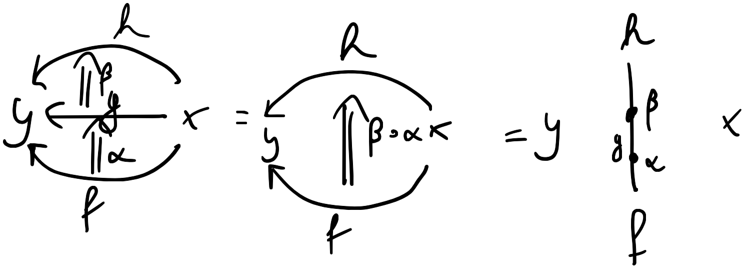


↳ horizontal composition:

globular diagram



↳ vertical composition:



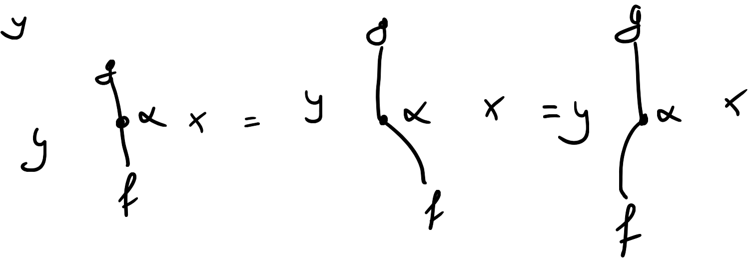
↳ 1_f : $1_g * \alpha = \alpha * 1_f = \alpha$

↳ Interchange law: $(\delta \circ \gamma) * (\beta \circ \alpha) = (\delta * \beta) \circ (\gamma * \alpha)$

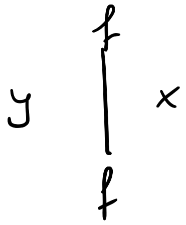
$\alpha : f \Rightarrow g$ $\qquad f : x \to y$

$\shortparallel$

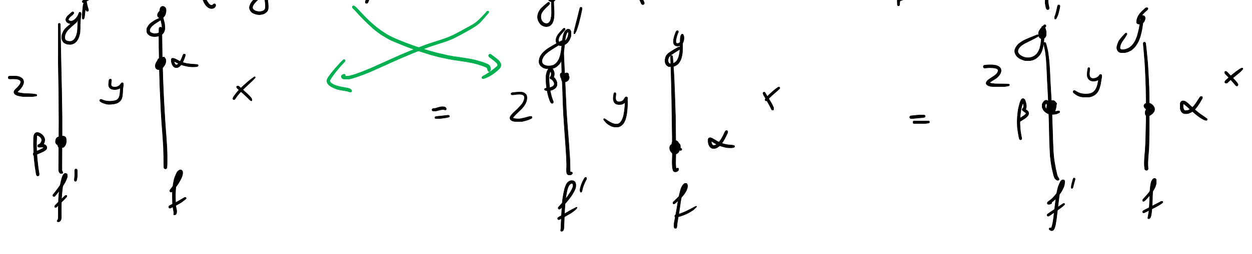
$1_y \circ f = f \circ 1_x$



$1_f : f \Rightarrow f$



$(\beta \circ 1_{f'}) * (1_g \circ \alpha) = (1_{g'} \circ \beta) * (\alpha \circ 1_f) = \beta * \alpha$



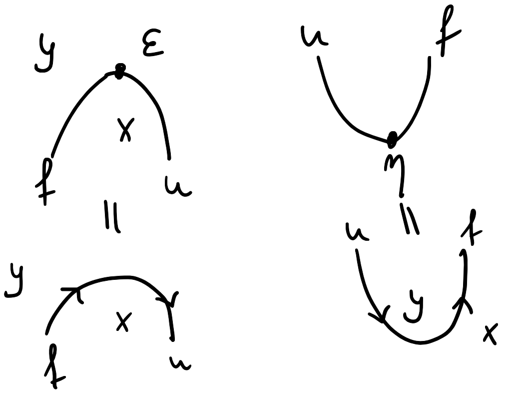
Biadjoints:

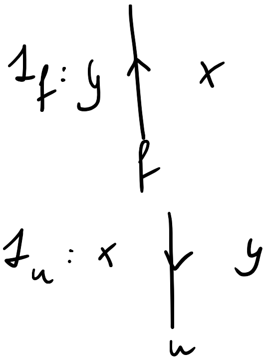
$f: x \to y$, $u: y \to x$ f is left adjoint to u: $\exists\ \overset{\text{counit}}{\varepsilon}: f\circ u \Rightarrow 1_y$

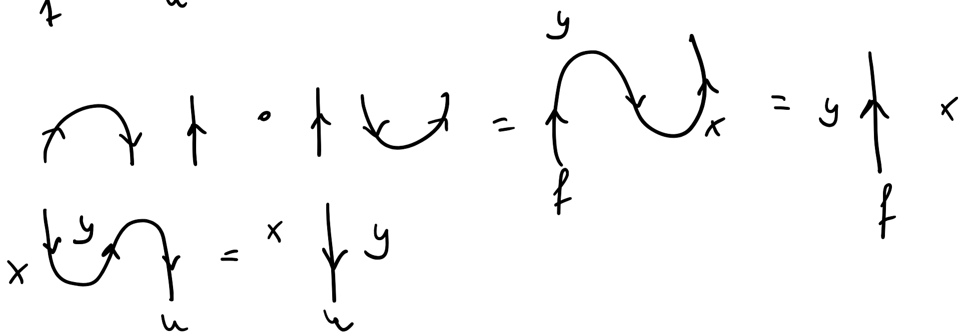
$\underset{\text{unit}}{\eta}: 1_x \Rightarrow u\circ f$

$(\varepsilon * 1_f)\circ(1_f * \eta) = 1_f$

$(1_u * \varepsilon)\circ(\eta * 1_u) = 1_u$

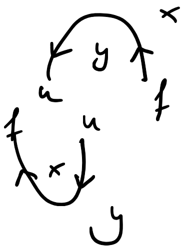


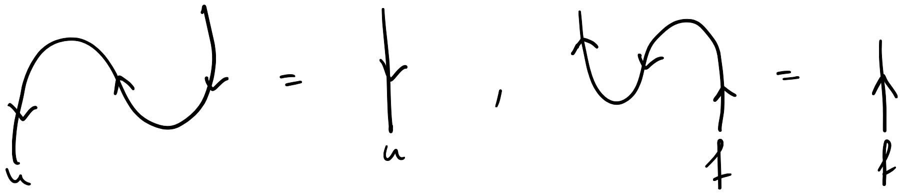




f is right adjoint to u: $\varepsilon': u\circ f \to 1_x$

$\eta': 1_y \to f\circ u$





Cat. of $\dot{U}$

$1_m \dot{U} 1_n$: elts = $\mathbb{Q}(q)$-lin. comb. of $1_m E_{(\varepsilon)} 1_n$

↳ $+ \Rightarrow \oplus$

↳ $\cdot q^{\pm 1} \Rightarrow \mathbb{Z}$-grading

$\mathcal{U}$: Objects: $n \in \mathbb{Z}$

1-morphisms: $\mathrm{Hom}_{\mathcal{U}}(n,m)$ $1_m \mathcal{E}_{(\varepsilon)} 1_n$

$\mathcal{E}_{(\varepsilon)} = \mathcal{E}_{\varepsilon_1}\cdots\mathcal{E}_{\varepsilon_m}$, $\mathcal{E}_+ = \mathcal{E}$, $\mathcal{E}_- = \mathcal{F}$

General elt of $\mathrm{Hom}_{\mathcal{U}}(n,m)$:

$1_m \mathcal{E}_{(\varepsilon_1)} 1_n\{t_1\} \oplus 1_m \mathcal{E}_{(\varepsilon_2)} 1_n\{t_2\}$

$\oplus \cdots$

finite direct sums.

$\mathbb{Z}$-graded vector space: $V = \bigoplus_{n\in\mathbb{Z}} V_n$

↳ $v \in \bigoplus_{n=M}^{N} V_n \Rightarrow t\in\mathbb{Z}: v\{t\} \in \bigoplus_{n=M+t}^{N+t} V_n$

↳ $\mathrm{gdim}(V) = \sum_{t\in\mathbb{Z}} q^t \dim(V_t)$

2-morphisms: preserve degree of 1-morphism

↳ $f \in \mathrm{Hom}_{\mathcal{U}}(n,m)$ of degree t

$\alpha : f \Rightarrow g \qquad g \in \mathrm{Hom}_{\mathcal{U}}(n,m)$ of degree t

$\deg(1_m \mathcal{E}_{(\varepsilon)} 1_n) = 0$

$\deg(1_m \mathcal{E}_{(\varepsilon)} 1_n\{t\}) = t$

$f\{t\}\{t'\} = f\{t+t'\}$

${}_n\mathcal{U}_m$ = Category with Objects = $\mathrm{Hom}_{\mathcal{U}}(n,m)$

$1\text{-Morph}(f,g) = 2\mathrm{Morph}_{\mathcal{U}}(f,g)$

$\mathcal{U}$ is categorification of $\dot{\mathcal{U}}$ under requirements:

- $K_0({}_n\mathcal{U}_m) = \underline{1_n \dot{\mathcal{U}} 1_m} \longrightarrow = \mathbb{Z}[q,q^{-1}]$-module

  ‖

  split Grothendieck group = $\{[f] : f \in \mathrm{Ob}({}_n\mathcal{U}_m)\}$

  ↳ $[f] = [f_1] + [f_2]$ if $f = f_1 \oplus f_2$
- $\mathbb{Z}[q,q^{-1}]$-module structure must be compatible with grading shifts:

  $[f\{t\}] = q^t [f]$
- ${}_n\mathcal{U}_m$ = additive category:
  - $\forall f,g \in \mathrm{Ob}({}_n\mathcal{U}_m)$: $\mathrm{Hom}_{{}_n\mathcal{U}_m}(f,g)$ is abelian group
  - Composition in $\mathrm{Hom}_{{}_n\mathcal{U}_m}$ is bilinear: $\alpha_1, \alpha_2 \in \mathrm{Hom}_{{}_n\mathcal{U}_m}(f,g)$, $\beta_1, \beta_2 \in \mathrm{Hom}_{{}_n\mathcal{U}_m}(g,h)$

    $\beta_1 \circ (\alpha_1 \cdot \alpha_2) = (\beta_1 \circ \alpha_1) \cdot (\beta_1 \circ \alpha_2)$

    $(\beta_1 \cdot \beta_2) \circ \alpha_1 = (\beta_1 \circ \alpha_1) \cdot (\beta_2 \circ \alpha_1)$
  - $f_1 \oplus \dots \oplus f_m \in \mathrm{Ob}({}_n\mathcal{U}_m)$ universally defined
- $\mathcal{U}$ = additive 2-category

  $f, f' \in 1\mathrm{Morph}_{\mathcal{U}}(n,m)$, $\quad g, g' \in 1\mathrm{Morph}_{\mathcal{U}}(m,p)$

  $g \circ (f \oplus f') = (g \circ f) \oplus (g \circ f')$

  $(g \oplus g') \circ f = (g \circ f) \oplus (g' \circ f)$
- $K_0(\mathcal{U}) = \underline{\bigoplus_{m,n \in \mathbb{Z}} K_0({}_n\mathcal{U}_m)}$ must satisfy $f = f_1 \circ f_2 \Rightarrow [f] = [f_1] \underset{\text{product in } \dot{\mathcal{U}}}{\cdot} [f_2]$

$\Downarrow$

Composition of 1-morphisms in $\mathcal{U}$ = multiplication in $\dot{\mathcal{U}}$

Remark: ① ${}_m\mathcal{U}_n$ only categorify $1_m \, {}_{\mathcal{A}}\dot{U} \, 1_n$, where ${}_{\mathcal{A}}\dot{U}$ is $\mathbb{Z}[q,q^{-1}]$-

subalgebra of $\dot{U}$ generated by divided powers:

$$\frac{E^a}{[a]!}1_n, \quad \frac{F^b}{[b]!}1_m, \quad a, b \in \mathbb{N}$$

$1_m \dot{U} 1_n = \mathbb{Q}(q)$-module $\hookleftarrow 1_m \, {}_{\mathcal{A}}\dot{U} \, 1_n = \mathbb{Z}[q,q^{-1}]$-module

↳ $\dot{\mathcal{U}}$ = cat. of ${}_{\mathcal{A}}\dot{U}$

② Lusztig's canonical basis of ${}_{\mathcal{A}}\dot{U}$ $\{b_x\}_x$: $b_x \cdot b_y = \sum_z m^z_{xy} b_z$

↳ $\in \mathbb{N}[q,q^{-1}]$

Indecomposable 1-morph. $\tilde{b}_x \Rightarrow \{[\tilde{b}_x]\}_x$ give basis for $K_0(\dot{\mathcal{U}})$

$[\tilde{b}_x][\tilde{b}_y] = \sum_z \tilde{m}^z_{xy} [\tilde{b}_z]$

↳ $\in \mathbb{N}[q,q^{-1}]$ (bc. $K_0(\dot{\mathcal{U}})$ graded)

$\Downarrow$

Indecomp. 1-morph. in $\dot{\mathcal{U}}$ = Lusztig can. basis elts (up to grading shift)

↳ $f \neq f_1 \oplus f_2$

↳ Which 1-morph. are indecomp.? 2-morph!

↳ 2-morph $\Rightarrow$ isomorphisms between 1-morph. $\rightsquigarrow$ $\dot{U}$-relations

Say $f, g \in \mathrm{Hom}_{\mathcal{U}}(n,m)$, $\mathcal{U}(f,g) := \{2\text{-morphisms } \alpha : f \Rightarrow g\}$

↳ vector space

$$\mathrm{HOM}_{\mathcal{U}}(f,g) = \bigoplus_{t \in \mathbb{Z}} \mathcal{U}(f\{t\}, g)$$

↓ $\mathbb{Z}$-graded vector space

$$\begin{tikzcd} \mathrm{HOM}_{\mathcal{U}}(\cdot,\cdot) : \arrow[d, Rightarrow, "\text{decategorify}"] & 1\mathrm{Morph}_{\mathcal{U}} \arrow[d, "K_0"] & \times & 1\mathrm{Morph}_{\mathcal{U}} \arrow[d, "K_0"] \arrow[r] & \mathrm{GrVect} : f \times g \to \mathrm{HOM}_{\mathcal{U}}(f,g) \arrow[d, "\mathrm{gdim}"] \\ \langle \cdot , \cdot \rangle & \dot{U} & \times & \dot{U} \arrow[r] & \mathbb{Z}[q,q^{-1}] \end{tikzcd}$$

$$\underline{\langle [f],[g] \rangle} = \mathrm{gdim}(\mathrm{HOM}_{\mathcal{U}}(f,g)) = \sum_{t \in \mathbb{Z}} q^t \underline{\dim(\mathcal{U}(f\{t\}, g))}$$

$$\underbrace{\langle [f\{t\}], [g] \rangle}_{\langle q^t[f],[g]\rangle} = \sum_{t'\in\mathbb{Z}} q^{t'} \dim(\mathcal{U}(f\{t+t'\}, g))$$

$t'' = t + t'$

$$= q^{-t} \sum_{t''\in\mathbb{Z}} q^{t''} \dim(\mathcal{U}(f\{t''\}, g)) = q^{-t} \langle [f],[g]\rangle$$

$$\underbrace{\langle [f], [g\{t\}] \rangle}_{\langle [f], q^t[g]\rangle} = q^t \langle [f],[g]\rangle$$

$\Rightarrow$ $\langle \cdot , \cdot \rangle$ is sesquilinear form on $\dot{U}$, $\mathbb{Z}[q,q^{-1}]$ - linear in 2^{nd} comp.
- antilinear in 1^{st} comp.

This form already exists!

↳ • $\langle 1_n x 1_m, 1_{n'} y 1_{m'} \rangle \neq 0 \Rightarrow n = n', \; m = m'$

$x, y \in U_q(sl_2)$

↳ $\mathrm{HOM}(f,g) \neq \{0\} \Rightarrow n = n' \wedge m = m'$

$\overset{\mathcal{U}}{f} \in 1\underset{\mathcal{U}}{\mathrm{Morph}}(m,n), \; g \in 1\mathrm{Morph}(m',n')$

• $\langle ux, y \rangle = \langle x, \tau(u) y \rangle \qquad x, y \in \dot{U}, \; u \in U_q(sl_2)$

↳ antilin. antiautomorph. on $\dot{U}$

• $[\langle E^a 1_n, E^a 1_n \rangle = \langle F^a 1_n, F^a 1_n \rangle = [a]!^2 \cdot \prod_{j=1}^{a} \frac{1}{1-q^{2j}}$.