Categorification of $\mathbb{Z}[S_n]$

What are we categoryfying ?

We want to categorify the regular representation of $\mathbb{Z}[S_n]$.
So, a categorification of $(\mathbb{Z}[S_n], a, \mathbb{Z}[S_n])$.

$$\begin{tikzcd}
K_0[\mathcal{B}] \arrow[r, "{[F_i]}"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\circlearrowright"] & K_0(\mathcal{B}) \arrow[d] \\
\mathbb{Z}[S_n] \arrow[r, "a_i"] & \mathbb{Z}[S_n]
\end{tikzcd}
\quad + \quad F_i F_j = \bigoplus_k F_k^{c_{ij}^k}$$

fM)

Where ?

We take our category $\mathcal{B}$ to be a block of the category $\mathcal{O}$ of $\mathfrak{sl}_n = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ with $\mathfrak{b}^+ = \mathfrak{n}^+ \oplus \mathfrak{h}$ the Borel algebra,

$\mathfrak{h}$ = cartan subalgebra
$\hookrightarrow$ traceless diagonal matrices

$\mathfrak{n}^+$ = strict upper triangular matrices
$\mathfrak{n}^-$ = strict lower triangular matrices.

For a weight $\lambda \in \mathfrak{h}^*$, we have a vector v_λ of the weight space.
The category $\mathcal{O}$ is spanned by Verma module

$$M(\lambda) = \mathcal{U}(\mathfrak{sl}_n) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C} v_\lambda$$

in the sense that any $M \in \mathcal{O}$ has a filtration of
$\hookrightarrow M_1 \hookrightarrow M_2 \hookrightarrow \cdots \hookrightarrow M_n = M$
with $M_i / M_{i-1} \simeq M(\lambda)$

Furthermore, we saw that $\mathcal{O}$ has enough projective so we note:

$L(\lambda)$ the simple quotient of $M(\lambda)$ and $P(\lambda)$ its principal module.

$$P(\lambda) \longrightarrow M(\lambda) \longrightarrow L(\lambda) \longrightarrow 0.$$

~~All the simple~~ module of M are isomorphic to one of $L(\lambda)$. Furthermore,

$$\mathrm{Hom}(P(\lambda), L(\mu)) = \begin{cases} \mathbb{C} & \lambda = \mu; \\ 0 & \text{else.} \end{cases}$$

The dot action and central characters

Note $\rho = \frac{1}{2}\sum_{\alpha > 0} \alpha$, the half-sum of positive roots.

the dot action $\cdot$ of S_n is

$$w \cdot \lambda = w(\lambda + \rho) - \rho.$$

We denote orbit of the dot action by $\mathfrak{h}^* / \cdot S_n$

Consider $Z(\mathfrak{gl}_n)$ the center of $\mathcal{U}(\mathfrak{gl}_n)$.

On $M(\lambda)$ for $z \in Z(\mathfrak{g})$ and $h \in \mathfrak{h}$ with maximal vector v^+

$$h \cdot z \cdot v^+ = z \cdot h \cdot v^+ = z \cdot \lambda(h) v^+ = \lambda(h) z \cdot v^+$$

then $\underset{z \in Z(\mathfrak{g})}{z \cdot v^+} = \chi_\lambda(z) v^+$ because the weight space M_λ has dimension 1.

$\chi_\lambda : Z(\mathfrak{sl}_n) \longrightarrow \mathbb{C}$ with kernel a maximal ideal of $Z(\mathfrak{sl}_n)$

is the central character of λ:

We saw that blocks of $\mathcal{O}$ divided with the central character because $\chi_\lambda = \chi_\mu$ if λ and μ have the same orbit under the dot action

and that then

$$\mathcal{O} = \bigoplus_{\nu \in \mathfrak{h}^* / \cdot S_n} \mathcal{O}_\nu .$$

All the blocks do not communicate $\mathrm{Ext}^i_{\mathcal{O}}(M, M') = 0$ $\forall i$ $M \in \mathcal{O}_\nu$ $M' \in \mathcal{O}_{\nu'}$ ν and ν' not in same orbits..

$\mathcal{O}_\nu$: M ~~composition ... with ...~~ $[M : L(\lambda)] = \{ 0$ if $\lambda \notin \nu$.

Regular blocks

An $\cdot$-orbit ν is *generic* if $w \cdot \lambda - \lambda$ is non integral $w \neq 1$.

$\mathcal{O}_\nu$ is then simply $n!$ $\mathbb{C}$-Vect.. copies.

An orbit is integral if $\nu \subset \Lambda^+$, subset of the positive weight lattice of $\mathfrak{h}^*$.

$\mathrm{Stab}(x) := \{ g \in G \mid g(x) =$

$\mathcal{O}_\nu$ is then indecomposable.

<u>Prop</u> $\mathcal{O}_\nu \simeq \mathcal{O}_{\nu'}$ as *categories* if

$\mathrm{stab}\, \lambda = \mathrm{stab}\, \lambda'$.

If $\mathrm{stab}\, \nu = \emptyset$ then $\mathcal{O}_\nu$ is called <u>regular block</u>

All regular blocks are equivalent

The block $\mathcal{O}_0$ containing the trivial module $L(0)$ of $\mathfrak{sl}_n$. Under the (shifted) action, it has $n!$ simple modules $L(w)$, $L(w \cdot 0)$.

So $K(\mathcal{O}_0)$ is abelian of rank $n!$.

We have two more basis of $K(\mathcal{O}_0)$, the Verma module $M(w)$ and $P(w)$.

Projective functors

Translation functors that we saw last week make possible the switch between blocks.

Translation through the i-th wall are functors

$\theta_i = \mathcal{O}_0 \longrightarrow \mathcal{O}_0$ given by

$\theta_i M(e) \simeq P(s_i)$. So $\theta_i(w) = P(w s_i \cdots$ are a generating set of projective endofunctors.

Because $P(w) \hookrightarrow M(w) \rightarrow 0$.

We have that θ_i acts on the Grothendieck group

$[\theta_i][M(w)] = [\theta_i(M(w))] = [P(w s_i)]$

$= M(w) + M(w s_i)$

Check for $\theta_i M e = [P(s_i)]$

$0 \rightarrow M(e) \xrightarrow{\theta_i} P(s_i) \longrightarrow M(s_i) \longrightarrow 0$

So we have an unique isomorphism.

$$\varphi : K(\mathcal{O}_0) \longrightarrow \mathbb{Z}[S_n]$$
$$[M(w)] \longmapsto w.$$

Put $C_w := \varphi([P(w)])$. C_w, $w \in W$ is a basis of $\mathbb{Z}[S_n]$.

For θ_i is thus lifted through φ to $1+s_i$.

indeed we have.

$$\theta_i^2 \simeq \theta_i \oplus \theta_i$$

$$\theta_i\theta_j \simeq \theta_j\theta_i \quad |i-j|>1$$

$$\theta_i\theta_{i+1}\theta_i \oplus \theta_{i+1} \simeq \theta_{i+1}\theta_i\theta_{i+1} \oplus \theta_i$$

$[\theta_i\theta_i]([M(w)]) = [\theta_i]\,[M(w)+M(ws_i)]$

$= [M w + M ws_i + M ws_i + M ws_i s_i]$

$= M w + M ws_i + M(w) + M(ws_i)$

$= \theta_i \oplus \theta_i\,(M(w))$

$|i-j|>1$

$$[\theta_i\theta_j](M(w)) = \theta_i M w + \theta_i M ws_j$$
$$= \theta_i M w + M ws_i + M ws_j + M w\underbrace{s_i s_j}_{= s_j s_i \quad |j-i|}$$
$$= \theta_j\theta_i M w.$$

and similarly for the last ones

$$\theta_i\theta_{i+1}\theta_i \simeq \theta_{s_i s_{i+1} s_i} \oplus \theta_i$$
$$\theta_{i+1}\theta_i\theta_{i+1} \simeq \theta_{s_{i+1}s_i s_{i+1}} \oplus \theta_{i+1}.$$

We thus have the first part of a categorification.

the $C_w := \varphi([\Omega(w)])$ are a basis:

$$C_w C_{w'} = \sum_{w''} c_{w'w}^{w''} C_{w''} \qquad c_{ww'}^{w''} \in \mathbb{Z}_{\geq 0}.$$

this comes from the decomposition of.

$$\Theta_w = \bigoplus \Theta_{i_k} \oplus \cdots \oplus \Theta_{i_1} \qquad w = s_{i_1} \cdots s_{i_k}$$

so it should work

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then composition projective functor will decompose as a direct sum of $[\Theta_w]$. and

$$\Theta_w \Theta_{w'} = \bigoplus_{w''} \Theta_{w''}^{c_{ww'}^{w''}} \qquad c_{ww'}^{w''} \in \mathbb{Z}_{\geq 0}.$$

We then have a weak abelian categorification