Category $\mathcal{O}$, Preriquisites

A word on sources

This small talk will go over the minimal concepts required to understand the categorification of $\mathbb{Z}[S_n]$, the group algebra of S_n. It uses the category $\mathcal{O}$ of Bernstein, Gel'fand and Gel'fand of $\mathfrak{sl}_n$. We follow the categorification presented in Khovanov, Mazorchuk, Stroppel (TAC) 2009. The present is given by the succinct treatment of Humphreys on BGG category.
We will make references to them as follow:

[H] : Representation of semisimple Lie algebras in the BGG category $\mathcal{O}$, 2008 AMS
[KMS]: A Brief review of abelian categorification Theo. Appl. Categories Vol 22, 2009, 479-508.
[H78] ~~Representation~~ of Lie algebras and rep theory GTM 9 Springer 1978

Lie algebras, classical approach

The treatment of Lie algebra in the classical way is now well known. We will remind two results here. First the triangular decomposition of $\mathfrak{sl}_n$ and the Poincaré Birkhoff Witt (PBW) basis theorem. All should work with semisimple Lie algebras

Consider $\mathfrak{g} = \mathfrak{sl}_n^{\text{over } \mathbb{C}}$. The root system of $\mathfrak{sl}_n$ is A_{n-1} with Weyl group S_n of rank $n-1$. Note Φ the root system of $\mathfrak{g}$, and denote, ~~for each α~~ the Cartan subalgebra of $\mathfrak{g}$ by $\mathfrak{h}$, it is the maximal nilpotent self-normalizing subalgebra of $\mathfrak{g}$. $\Phi \in \mathfrak{h}^*$

$\exists n,\ ad_x^n h = 0 \ \ \forall x \in \mathfrak{h}, h \in \mathfrak{h}$ $\quad$ $[X,Y] \in \mathfrak{h} \ \forall X \in \mathfrak{h} \Rightarrow Y \in \mathfrak{h}$

For each root $\alpha \in \Phi$, we denote the root space associated to it by

$$\mathfrak{g}_\alpha := \{ x \in \mathfrak{g} \mid [h, x] = \alpha(h) x,\ h \in \mathfrak{h} \}.$$

It is of dimension 1. From this we have the triangular decomposition of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+, \qquad \mathfrak{n}^+ := \bigoplus_{\alpha>0} \mathfrak{g}_\alpha, \quad \mathfrak{n}^- := \bigoplus_{\alpha<0} \mathfrak{g}_\alpha$$

For $\mathfrak{sl}_n$, $\mathfrak{n}^-$ is the subalgebra of strictly lower diagonal matrices, $\mathfrak{n}^+$, that of strictly upper diagonal, and $\mathfrak{h}$ of trace zero diagonal matrices, spanned by

$E_i - E_{i+1}$ for $E_i = (e_{ii} = 1, \text{rest} = 0)$

Two maximal solvable Lie subalgebras

$\leadsto ad_{\mathfrak{b}}^n[\mathfrak{b}] = 0$ for some n

$\mathfrak{b}^+ := \mathfrak{n}^+ \oplus \mathfrak{h}$ and $\mathfrak{b}^- = \mathfrak{n}^- \oplus \mathfrak{h}$

are the Borel subalgebras.

Recall $ad_x(y) = [x, y]$, the adjoint representation.

It is always useful to study the universal enveloping algebra of $\mathfrak{g}$, denoted $\mathcal{U}(\mathfrak{g})$, an associative unital algebra, infinite dimensional.

It is universal in the sense that for any map $\mathfrak{g} \xrightarrow{\varphi} A$ to an associative algebra A, we have a unique morphism $\hat{\varphi}$ making the algebra diagram commute

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "h"] \arrow[dr, "\varphi"'] & U(\mathfrak{g}) \arrow[d, dashed, "\hat{\varphi}"] \\ & A \end{tikzcd}$$

Any basis of $\mathfrak{g}$ will give rise to one of $U(\mathfrak{g})$, and if it is ordered, the order will keep through. This is the Poincaré-Birkhoff-Witt (PBW) property. In particular for $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^-$, we will get a PBW basis of

$$U(\mathfrak{n})\, U(\mathfrak{h})\, U(\mathfrak{n}^-) = U(\mathfrak{g}).$$

Note by $\mathrm{Mod}\, U(\mathfrak{g})$ the category of $U(\mathfrak{g})$ modules (left). For $\lambda \in \mathfrak{h}^*$, denote: (for $M \in \mathrm{Mod}\, U(\mathfrak{g})$).

$$M_\lambda := \{ v \in M \mid h \cdot v = \lambda(h) v \quad \text{for all } h \in \mathfrak{h} \}$$

If it is not 0, λ is called a <u>weight</u> of M and its <u>multiplicity</u> is $\dim M_\lambda$. M_λ is the <u>weight space</u>.

Denote $Z(\mathfrak{g}) \subset U(\mathfrak{g})$ the center of $U(\mathfrak{g})$

Category $\mathcal{O}$

The category $\mathcal{O}$ is the full subcategory of $\mathrm{Mod}(\mathcal{U}(\mathfrak{g}))$ satisfying the three conditions for all $M \in \mathcal{O}$ ~~of Mod(U(g))~~

$\mathcal{O}_1$) M is finitely generated

$\mathcal{O}_2$) M is $\mathfrak{h}$-semisimple, so $M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$, it is a sum of weight modules, and thus also a weight module

$\mathcal{O}_3$) M is locally $\mathfrak{n}$-finite: for each $v \in M$, $\mathcal{U}(\mathfrak{n}^+) \cdot v \subset M$ is finite dimensional, so $\mathcal{U}(\mathfrak{n}^+)$ acts locally-nilpotently.

Already from these axiom we have two more.

$\mathcal{O}_4$) All weight spaces of M are finite dimensional.

$\mathcal{O}_5$). $\Pi(M) := \{\lambda \in \mathfrak{h}^* \mid M_\lambda \neq 0\}$ is contained in the union of finitely many cut $\lambda - \Gamma$, $\lambda \in \mathfrak{h}^*$ and Γ generated by ϕ^+, an root lattice.

Technical

prop $\mathcal{O}$ Here is a list of properties of $\mathcal{O}$.

a) $\mathcal{O}$ is a noetherian category, $M \in \mathcal{O}$ is noetherian.

b) $\mathcal{O}$ is closed under submodules, quotients and finite direct sums

c) $\mathcal{O}$ is abelian as a category

d_0) For L, a finite dimensional, $L \in \mathcal{O}$

d_1) $M \mapsto L \otimes M$ is an exact endofunctor of $\mathcal{O}$.

e) All $M \in \mathcal{O}$ are $Z(\mathfrak{g})$-finite: $\dim \{ z \cdot v \mid z \in Z(\mathfrak{g}) \} < \infty$ for all $v \in M$.

f) $M \in \mathcal{O}$ is finitely generated as $\mathcal{U}(\mathfrak{n}^-)$-module

The category $\mathcal{O}$ is often called the highest weight category for the reasons outlined in the following discussion

Let $v^+ \in M$, for a $\mathcal{U}(\mathfrak{g})$ module M. It is a <u>maximal vector</u> of weight $\lambda \in \mathfrak{h}^*$ if $v^+ \in M_\lambda$ and $\eta^+ \cdot v^+ = 0$.
From O2 and O3, all $M \in \mathcal{O}$ have at least one maximal vector.

We say that $M \in \text{Mod}(\mathcal{U}(\mathfrak{g}))$ is <u>a highest weight ~~vector~~ module</u> of weight λ if there exist a maximal vector $v^+ \in M_\lambda$ such that $M = \mathcal{U}(\mathfrak{g}) \cdot v^+$. PBW implies $M = \mathcal{U}(\eta^-) \cdot v^+$.

~~It is~~ All highest weight modules are in $\mathcal{O}$.

Prop highest weight module Let M be a highest weight module of weight $\lambda \in \mathfrak{h}^*$ with maximal vector v^+. ~~Order Φ^+ $\alpha_1 < \alpha_2 < \cdots < \alpha_m$ wit~~ Then.

a) $M \in \mathcal{O}$, and M_μ is finite dimensional for all weight μ.
b) Each quotient of M is a highest weight module with λ.
c) Each submodule of M is a weight module.
   A submodule generated by a maximal vector of weight $\mu < \lambda$ is proper.
d) M has a unique maximal submodule and unique simple quotient, so it is indecomposable
e) All simple highest weight module of weight λ are isomorphic and $\dim \text{End}_{\mathcal{O}} M = 1$ for simple h.w.M.

Highest weight module are the building block of $\mathcal{O}$ because

Coro Let $M \in \mathcal{O}$. Then M has a finite filtration

$0 \subset M_1 \subset M_2 \subset \cdots \subset M_n = M$

with ~~sp~~ non-zero quotients $M_i / M_{i-1} \cong N_i$ for highest weight modules N_i.

Verma modules

The **Verma modules** are obtained by induction.

Let $\mathfrak{b}$ be the Borel subalgebra and consider $\mathfrak{b}/\mathfrak{n}^+ \simeq \mathfrak{h}$.

For $\lambda \in \mathfrak{h}^*$, we have a 1-dimensional $\mathfrak{b}$ module ~~trivial~~ on $\mathfrak{n}$

denoted $\mathbb{C}_\lambda$ (or $\mathbb{C}_\lambda$).

Set $M(\lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_\lambda$, the Verma module.

The vector $v_+ = 1 \otimes 1$ is maximal module generating $M(\lambda)$.

Using properties of highest weight modules, write $L(\lambda)$ to be the unique simple quotient of $M(\lambda)$.

Theorem Every simple module of $\mathcal{O}$ is isomorphic to a module $L(\lambda)$ $\lambda \in \mathfrak{h}^*$ and is determined uniquely (up to iso) by its highest weight.

$$\dim \operatorname{Hom}_{\mathcal{O}}(L(\mu), L(\lambda)) = \delta_{\lambda\mu}.$$

There are a family of finite-dimensional simple modules:

Theorem $L(\lambda) \in \mathcal{O}$ is finite-dimensional if and only if $\lambda \in \Lambda^+$ also if and only if

$$\dim L(\lambda)_\mu = \dim L(\lambda)_{w\mu}, \quad \forall \mu \in \mathfrak{h}^*, \ w \in W$$

$\Lambda_+ := \mathbb{Z}^+ w_1 + \cdots + \mathbb{Z}^+ w_\ell$, $\langle w_i, \alpha_j^\vee \rangle = \delta_{ij}$, fundamental weight. $\to$ dominant integral weight.

What more to say?

To say more of the category $\mathcal{O}$, we need more information, not just the PBW ~~stefor~~ basis. For more information we need information coming from the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$.

This is done via the Harish-Chandra homomorphism

$$pr : U(\mathfrak{g}) \to U(\mathfrak{h}) \quad , \quad \xi = pr|_{Z(\mathfrak{g})}.$$

This is used to know things from the components in

Prop a) the category $\mathcal{O}$ is artinian

b) $M \in \mathcal{O}$ is of finite composition, or Jordan-Hölder length.

So $0 \hookrightarrow M_0 \hookrightarrow M_1 \subset \cdots M_m \hookrightarrow M$ with $M_i / M_{i-1} \cong L(\mu)$.

A consequence of being artinian and noetherian, each M is writable as a direct sum of indecomposable morphism (Krull - Schmidt).

Therefore, $K(\mathcal{O})$, the Grothendieck group is free Abelian group in $\{[L(\lambda)] \mid \lambda \in \mathfrak{h}^*\}$ and

$$K(M) = \sum_{\lambda \in \mathfrak{h}^*} \underbrace{[M : L(\lambda)]}_{\text{composition multiplicity of } L(\lambda) \text{ in } M} [L(\lambda)]$$

Vocabulary:

<u>socle</u> $\mathrm{Soc}(M)$: $\oplus$ simple submodule M

<u>radical</u> $\mathrm{Rad}(M)$: $\cap$ maximal submodule

head $\mathrm{Hd}(M)$: $M / \mathrm{Rad}\, M$

o

Projective modules, are there enough?

A projective module $P \in \mathcal{O}$ is such that $\mathrm{Hom}(P, -)$ is exact functor. Because $\mathcal{O}$ is artinian, we could study its representation theory via the principal modules (so the indecomposable projective modules) if the category has enough projective. First we give a big family of modules projective and then state the theorems

Prop [38]
a) if λ is dominant, and so maximal in its $W_{[\lambda]}$-orbit then $M(\lambda)$ is projective
b) if $P \in \mathcal{O}$ is projective ~~and~~ for L such that $\dim L < \infty$ then $P \otimes L$ is projective.

Theorem $\mathcal{O}$ has enough projective and injective

Idea of proof
$\lambda \in \mathfrak{h}^*$.
$\mu := \lambda + n\rho$ is dominant for large n. $\rho := \omega_1 + \dots + \omega_l \in \Lambda$ half sum of positive roots
then prop $\Rightarrow$ $M(\mu)$ proj. $n\rho \in \Lambda^+$ so $\dim L(n\rho) < \infty$. $P := M(\mu) \otimes L(n\rho)$ is also projective.
So $M(\mu - n\rho) = M(\lambda)$ is a quotient of $M(\mu)$ — by verma module tensoring process.
So $M(\lambda)$, and $L(\lambda)$ is a quotient of a projective!

From this, ~~each~~ $M \in \mathcal{O}$ have a projective cover $P_M \xrightarrow{\pi_M} M$. Up to isomorphism.
Note $P(\lambda)$ the projective cover of $L(\lambda)$, it is also that of $M(\lambda)$. It is well-defined

Theorem $\{P(\lambda) \mid \lambda \in \mathfrak{h}^*\}$ respect.
a) Every indecomposable module is isomorphic to some $P(\lambda)$.
b) if $P = \bigoplus P(\lambda) c_\lambda$, the coeff c_λ are $c_\lambda = \dim \mathrm{Hom}_{\mathcal{O}}(P, L(\lambda))$
c) $\dim \mathrm{Hom}_{\mathcal{O}}(P(\lambda), M) = [M : L(\lambda)]$
in particular $\dim \mathrm{Hom}_{\mathcal{O}} (P(\lambda), L(\mu)) = \delta_{\lambda\mu}$.

We can have a filtration of module called <u>standard filtration</u> or <u>Verma flag</u>

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_n = M$$

With each $M^i = M_i / M_{i+1}$ isomorphic to a Verma module $M(\lambda)$

We note the multiplicity of $M(\lambda)$ by $(M : M(\lambda))$.

⚠ It is not the multiplicity of simple module ~~L(λ)~~ noted $[M : L(\lambda)]$.

The two last theorem are the main result of BGG.

Theorem Each projective has a standard filtration and $(P(\lambda) : M(\mu)) \neq 0$ only if $\mu \geq \lambda$ and $(P(\lambda) : M(\lambda)) = 1$.

Theorem BGG reciprocity Let $\lambda, \mu \in \mathfrak{h}^*$.

$$(P(\lambda) : M(\mu)) = [M(\mu) : L(\lambda)] = [M(\mu)^\vee : L(\lambda)].$$

For example For maximal λ, $P(\lambda) = M(\lambda)$.

Block

In analogy with the finite groups case studied by Brauer, we say that block separate the simple modules by putting M_1 and M_2 in the same block if there exists ~~a~~ non-split exact sequence

$$0 \longrightarrow M_1 \longrightarrow M \longrightarrow M_2 \longrightarrow 0$$

$$0 \longrightarrow M_2 \longrightarrow M \longrightarrow M_1 \longrightarrow 0$$

M is in a block if all its simple factors are in.

An arbitrary M will decompose as direct sum of ~~block~~ submodules each belonging to a single block. So an indecomposable ~~blockdule~~ belongs to a single block.

Humphreys describes the category as a sum of blocks via central characters. The block $\mathcal{O}_0$ is called the principal block attached to the character of 0.

Each block is isomorphic to a category of finite dimensional modules of a $\mathbb{C}$-algebra of finite-dimensional. Hard to describe explicitly.

Orbits for a element are generic or integral depending of the action of W on weight is integral or not. Generic orbit leads to boring ~~block~~ for $\mathfrak{g} = \mathfrak{sl}_n$. and such a have $P(\lambda) = M(\lambda) = L(\lambda)$.

All regular blocks, block for which the stabilizer is trivial, are isomorphic. So we can just study the principal block containing $L(0)$, the 1d trivial rep.

Translation functors.

We enter the last part of our subject. It is chapter 7 of Humphreys. We present some results and hide some technical details under the rug.

This constitutes the starting point of Kazhdan-Lusztig theory.

Def Fix $\lambda, \mu \in \mathfrak{h}^*$ two weights and let $pr_\lambda : \mathcal{O} \longrightarrow \mathcal{O}_{\chi_\lambda}$ and $pr_\mu : \mathcal{O} \longrightarrow \mathcal{O}_{\chi_\mu}$ be the natural projection. Let $\mu - \lambda \in \Lambda$ so μ and λ are called <u>compatible</u>. The orbit of $\nu = \mu - \lambda$ by W contains a unique weight $\bar{\nu}$ in Λ^+. Put $L = L(\bar{\nu})$.

The translation functor T_λ^μ is the exact functor defined as:

$$T_\lambda^\mu : \mathcal{O}_{\chi_\lambda} \longrightarrow \mathcal{O}_{\chi_\mu}$$

$$M \longmapsto pr_\mu(L \otimes (pr_\lambda M)).$$

or

$$T_\lambda^\mu : \mathcal{O} \longrightarrow \mathcal{O}$$

$$M \longmapsto inc_\mu(pr_\mu(L \otimes (pr_\lambda M))).$$

Properties

a) $T_\lambda^\mu (M^\vee) \simeq (T_\lambda^\mu M)^\vee$.

b) projectives are sent to projectives

c) For all $M, N \in \mathcal{O}$

$$\mathrm{Hom}_\mathcal{O}(T_\lambda^\mu M, N) \simeq \mathrm{Hom}_\mathcal{O}(M, T_\mu^\lambda N)$$

$$\mathrm{Ext}^n_\mathcal{O}(T_\lambda^\mu M, N) \simeq \mathrm{Ext}^n_\mathcal{O}(M, T_\mu^\lambda N)$$

d) T_λ^μ induces an endomorphism on Grothendieck group.

$\mathrm{Ext}^n_\mathcal{O}(M, N) = \mathrm{Ker}\,\mathrm{Hom}_\mathcal{O}(\cdot h_{n+1}, N) / \mathrm{Im}\,\mathrm{Hom}_\mathcal{O}(h_n, N)$.
for a projective resolution $P_\cdot$ of M.

$$\leftarrow \mathrm{Hom}(P_1, N) \xleftarrow{\mathrm{Hom}(h_1, N)} \mathrm{Hom}(P_0, N) \leftarrow \mathrm{Hom}(M, N)$$

$$\Uparrow \mathrm{Hom}(-, N)$$

$$P_2 \longrightarrow P_1 \xrightarrow{h_1} P_0 \to M \to 0$$

We now skip non integrable discussion and combinatorics of Weyl groups.

Theorem (Assume certain technical condition on weight).
$\rightarrow$ The translation functor on Verma modules. $\leftarrow$

$$T_\lambda^\mu M(w\cdot\lambda) \simeq M(w\cdot\mu).$$

for the "dot" action on W. $\rightarrow$ in S_n $w\cdot\lambda = w(\lambda+\rho)-\rho$ shifted by half-sum of positive roots.

M a conv. $M \in \mathcal{O}_\lambda$ (lemma) a standard filtration implies that $T_\lambda^\mu M$ has a standard filtration in $\mathcal{O}_\mu$.

Theorem (Assume certain condition).
$\longrightarrow$ Translation functor on simple module. $\leftarrow$

$$T_\lambda^\mu L(w\cdot\lambda) = \begin{cases} 0 \\ \simeq L(w\cdot\mu) \end{cases}$$

Theorem (Assume certain condition)
$\longrightarrow$ Equivalence of category $\leftarrow$

T_λ^μ and T_μ^λ define equivalence of categories between $\mathcal{O}_\lambda$ and $\mathcal{O}_\mu$ and send simple to simple.

Condition on weight $\lambda, \mu \in \mathfrak{h}^*$ antidominant and compatible.
$\lambda^\#, \mu^\#$ lie in the same facet of $E(\lambda)$
$\hookrightarrow$ integral weight related to λ and μ

facet $\rightarrow$ subset determined by partition of Φ^+ in disjoint subsets
$\rightarrow$ see dJkv^W-

With certain condition that seem to hold for integral weights we have also projective equivalence:

Theorem

$$T_\mu^\lambda(P(w\cdot\mu)) \simeq P(w\cdot\lambda).$$

Translation through the wall

We consider now the <u>wall-crossing functor</u>
$\theta_s := T_\mu^\lambda T_\lambda^\mu$ where λ and μ are integral and lie in the s-wall of the chamber containing λ.

Theo

$$\theta_\lambda L(w\cdot\lambda) \simeq L(w\cdot\lambda) \oplus L(w\cdot\lambda)$$

$$\mathrm{Hom}_{\mathcal O}(T_\lambda^\mu M, T_\lambda^\mu M) \simeq \mathrm{Hom}_{\mathcal O}(M, T_\mu^\lambda T_\lambda^\mu M)$$

$$\theta_\lambda M(w_\lambda\cdot\lambda) \simeq P(w_\lambda\cdot\lambda\cdot\lambda).$$

classified

It can be used in the classification theorem of projective functors

theo Projective functors decompose ~~into~~ their restriction to a block $\mathcal O_\lambda$ into a direct sum of indecomposable functors

$F_{(\lambda,\mu)} : M(\lambda) \longrightarrow P(\mu) \longrightarrow$ ^Tr $\theta M(w$

$\hookrightarrow$ orbit under act. $(W\cdot\lambda, W\cdot\mu)$