Kleine seminar

3/11 Strong categorification

Enriched category C over M if
monoidal

- Hom spaces of C are objects in M
- $Mor_C(Y,Z) \otimes Mor_C(X,Z) \to Mor_C(X,Z)$
  $\delta \otimes \gamma \mapsto \delta \circ \gamma$

$\alpha \times \!\!\!/\, Z$
morphism in M

- preadditive category — enriched over abelian groups
- R-linear category — enriched over cat of R-modules
- 2-category — enriched over categories

associative unital ring

Algebraic objects		Category
• monoid	$\leftrightarrow$	category with one object
• group	$\leftrightarrow$	" " and every homomorphism is an isomorphism
• ring	$\leftrightarrow$	category with one object (preadditive)
• R-algebra	$\leftrightarrow$	R-linear cat with one object
• Ring B with idempotents $\{e_1, \dots e_n\}$	$\leftrightarrow$	preadditive cat with n objects $\{1, \dots, n\}$
$e_ie_j = 0$ if $i \neq j$		
$B_{ij} = e_i B e_j$		$Hom(i,j) = B_{ij}$
$B = \bigoplus_{i,j} B_{ij}$		

$1 = e_1$

Functor F from a group G to Set

$F(*)$ is a set

$F(\alpha)$ is an automorphism of $F(*)$

$\to$ action of our group on a set

Functor from G to $Vect_{\mathbb{F}}$ — group representation
monoid algebra

- Grothendieck group of a 2-category $\mathcal{C}$

$K_0^{split}(\mathcal{C})$: obj of $K_0(\mathcal{C})$ are object of $\mathcal{C}$

$$\mathrm{Mor}_{K_0(\mathcal{C})}(X, Y) = K_0^{split}(\mathrm{Mor}_{\mathcal{C}}(X, Y))$$

additive 2-categories : enriched over additive categories
abelian " " " abelian categories

<u>Def</u> Strong categorification

Let C be an R-linear category

A strong categorification of C is a pair $(\mathcal{G}, \varphi)$

- $\mathcal{G}$ is an additive 2-category
  $\varphi : K_0^{split}(\mathcal{G}) \otimes_{\mathbb{Z}} R \to C$ is an isomorphism
- $\mathcal{G}$ is an abelian 2-category
  $\varphi : K_0(\mathcal{G}) \otimes_{\mathbb{Z}} R \to C$ is an isomorphism

<u>Example</u>

$D = \mathbb{C}[x]/(x^2)$ $\quad X = D \otimes_{\mathbb{C}} D$ $\quad$ D-D bimodule

$\mathcal{G}$. 2-category

obj $\quad \underline{I} = D\text{-mod}$

$\mathrm{Mor}_{\mathcal{G}}(I, I)$: full additive subcategory of endofunctors isomorphic to direct sums of copies of $\begin{cases} id \\ F = X \otimes_D - \end{cases}$

So $\boxed{F \circ F \cong F \oplus F}$

$$X \otimes_D X = D \otimes_{\mathbb{C}} D \otimes_D D \otimes_{\mathbb{C}} D \cong D \otimes_{\mathbb{C}} D \otimes_{\mathbb{C}} D$$
$$\cong D \otimes_{\mathbb{C}} D \oplus D \otimes_{\mathbb{C}} D = X \oplus X$$

Consider $B = \mathbb{C}[b]/(b^2 - 2b)$

$$\varphi : K_0^{split}(\mathcal{G}) \otimes_{\mathbb{Z}} \mathbb{C} \to B$$
$$[F] \mapsto b$$
$$[id] \mapsto 1$$