27/10
2020

Kleine seminar

Strong categorification of the Heisenberg algebra

Based on Savage, introduction to categorification.

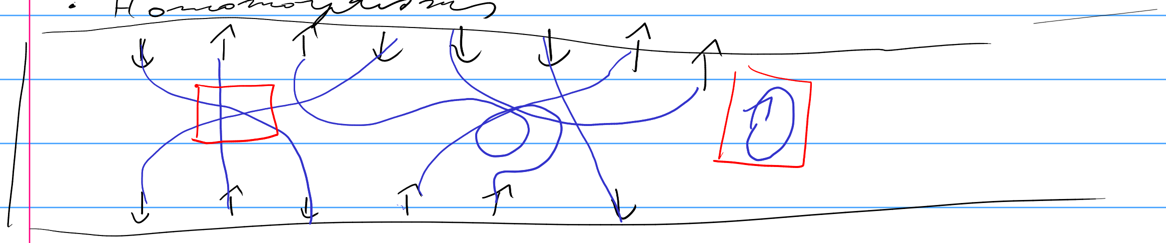
- Graphical Heisenberg category

Monoidal category $\mathcal{H}$

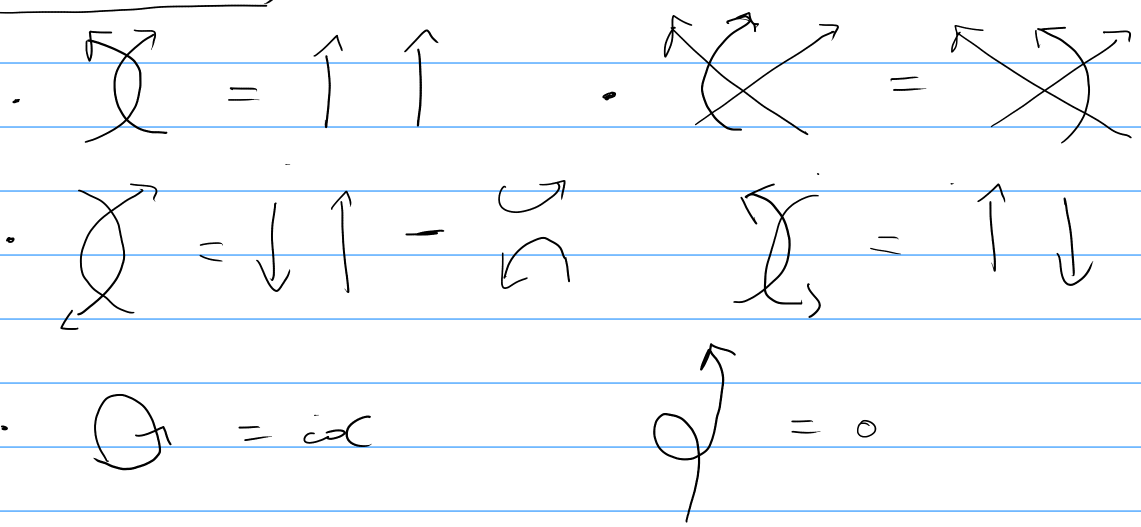
- objects : $\uparrow, \downarrow$, $\mathbb{1}$

$\uparrow \otimes \downarrow \otimes \uparrow \otimes \downarrow \otimes \downarrow$

- Homomorphisms



local relations



$e_n \;,\; h_n^* \quad n \in \mathbb{N} \qquad (e_n), (h_n^*)$ bases of Sym
Sym*

$$\begin{cases} [e_n, e_m] = [h_n^*, h_m^*] = 0 \\ [h_m^*, e_n] = e_{n-1} h_{m-1}^* \end{cases}$$

weak categorification

$\mathcal{A} = \bigoplus_{n \in \mathbb{N}} A_n\text{-mod} \qquad \varphi_A : K_0(\mathcal{A}) \to \text{Sym}$
$[S^\lambda] \mapsto s_\lambda$

$\text{Ind}_M : \quad \mathcal{A} \to \mathcal{A}$
$N \mapsto \text{Ind}_M(N) = \text{Ind}_{A_m \otimes A_n}^{A_{m+n}} M \otimes N$

$\text{Res}_M : \quad \mathcal{A} \to \mathcal{A}$
$N \mapsto \begin{cases} \text{Hom}_{A_m}(M, \text{Res}^{A_n}_{A_{n-m} \otimes A_m} N) & \text{if } n \geq m \\ 0 \end{cases}$

$[\text{Ind}_M] = \varphi_A([M])$
$[\text{Res}_M] = \varphi_A(M)^*$

$$\begin{cases} [\text{Ind}_{E_n}] = e_n \\ [\text{Res}_{L_n}] = h_n^* \end{cases}$$

$(A_m ; A_n)$

$F_n : \quad \mathcal{H}' \to \bigoplus_m {}_m\text{bimod}_n \qquad A_k \hookrightarrow A_{k+1}$

$\uparrow \quad \mapsto \quad {}_{(n+1)}(A_{n+1})_{(n)}$
$\downarrow \quad \mapsto \quad {}_{n-1}(A_n)_n$

$\uparrow \downarrow \uparrow \uparrow : \mapsto {}_{n+2}(A_{n+2}) \otimes_{n+1} (A_{n+2}) \otimes_{n+2} (A_{n+2}) \otimes_{n+1} (A_{n+1})_n$

${}_m\text{bimod}_n \to \text{Fun}(A_n\text{-mod}, A_m\text{-mod})$
$M \mapsto M \otimes_{A_n}$

$F_n : \mathcal{H}' \to \bigoplus_m {}_m\text{bimod}_n \to \text{Fun}(A_n\text{-mod}, \mathcal{A})$
$F : \mathcal{H}' \xrightarrow{\oplus_n F_n} \text{Fun}(\mathcal{A}, \mathcal{A})$
$\uparrow \mapsto \bigoplus_n \text{Ind}_{A_n}^{A_{n+1}} \cong \text{Ind}_{A_1}$
$\uparrow^{\otimes m} \mapsto \bigoplus_n \text{Ind}_{A_n}^{A_{n+m}} \cong \text{Ind}_{A_m}$

$F_n(\curvearrowright)$: ${}_n(A_n) \otimes_{n-1} (A_n)_n \longrightarrow {}_n(A_n)_n$

$x \otimes y \longmapsto xy$

$F_n(\circlearrowleft)$: ${}_n(A_n)_n \longrightarrow {}_{(n)}\overbrace{A_{n+1} \otimes_{(n+1)} A_{n+1}}^{A_{n+1}}{}_{(n)}$

$z \longmapsto z$

$F_n(\circlearrowright)$: ${}_{(n)}\underbrace{A_{n+1} \otimes_{n+1} (A_{n+1})_n}_{{}_n A_{n+1}{}_n} \longmapsto A_n$

$F_n(\curvearrowleft)$ $\quad x_i \longmapsto x_i$

$\quad x_n \longmapsto 0$

$F_n(\times)$

$F_n(\bigcirc) = {}_n A_n{}_n \to {}_n A_n{}_n = \mathrm{id}$.

$z \mapsto z \mapsto z$

$F: \mathcal{H}' \longrightarrow \mathrm{End}(\mathcal{C}(\mathcal{A}))$

$F(\uparrow^{\otimes m}) \longmapsto \mathrm{Ind}_{A_m}$

$F(\downarrow^{\otimes m}) \longmapsto \mathrm{Res}_{A_m}$

Every simple A_n-module is contained in decomposition series of A_n

Idempotent completion

$X \cong Y \oplus Z$

$e: X \twoheadrightarrow Y \hookrightarrow X$ idempotent

A category is idempotent complete if every idempotent has a corresponding subobject Y

Def Karoubi envelope (idempotent completion) of C

Obj: (X, e) $\quad X \in \mathrm{Obj}\, C$, e is idempotent $\in \mathrm{Hom}_C(X, X)$

Morph $\varphi: (X, e) \to (Y, f)$

$\varphi \in \mathrm{Hom}_C(X, Y)$

$f \circ \varphi \circ e = \varphi$

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "e"'] \arrow[dr, "\varphi"] & Y \arrow[d, "f"] \\ X \arrow[r, "\varphi"'] & Y \end{tikzcd}$$ commute

Graphical Heisenberg category

$\mathcal{H}$ = Karoubi envelope of $\mathcal{H}'$.

$$A_n \longrightarrow \mathrm{End}_{\mathcal{H}'}(\uparrow^{\otimes n})$$
$$A_n \longrightarrow \mathrm{End}_{\mathcal{H}'}(\downarrow^{\otimes n})$$

$$e(n) = \frac{1}{n!}\sum_{\sigma\in S_n}\sigma \qquad f(n) = \frac{1}{n!}\sum_{\sigma\in S_n}(-1)^{|\sigma|}\sigma \qquad \text{idempotents}$$

Obj in $\mathcal{H}$

$$\Lambda^n_\uparrow\ (\uparrow^{\otimes n}, f(n)) \ , \quad S^n_\downarrow = (\downarrow^{\otimes n}, e(n))$$

Prop

$$\begin{cases} S^n_\downarrow\otimes S^m_\downarrow = S^m_\downarrow\otimes S^n_\downarrow \ , \quad \Lambda^n_\uparrow\otimes\Lambda^m_\uparrow = \Lambda^m_\uparrow\otimes\Lambda^n_\uparrow \\ S^m_\downarrow\otimes\Lambda^n_\uparrow = \Lambda^n_\uparrow\otimes S^m_\downarrow \oplus \Lambda^{n-1}_\uparrow\otimes S^{m-1}_\downarrow \end{cases}$$

Theorem injective morphism as $\boxed{\mathbb{Z}\text{-algebras}}$ homomorphism

$$\varphi : H \longrightarrow K_0^{split}(\mathcal{H})$$
$$h^*_n \longmapsto [S^n_\downarrow]$$
$$e_n \longmapsto [\Lambda^n_\uparrow]$$

$\boxed{\text{Conjecture : } \varphi \text{ is isomorphism}}$

$$F(\Lambda^n_\uparrow) = \mathrm{Ind}_{E_n}$$
$$F(S^n_\downarrow) = \mathrm{Res}_{L_n}$$

$$\begin{tikzcd}
 & \mathcal{H} \arrow[r, "F"] \arrow[d, "K_0^{split}"'] & \mathrm{Fun}(\mathcal{A},\mathcal{A}) \arrow[d] \\
\mathrm{cat} & K_0(\mathcal{H}) \arrow[l] \arrow[r, "{[F]}"] & \mathrm{End}_{\mathbb{Z}}(\mathrm{Sym}) \\
\mathrm{cat} \arrow[u] & H \arrow[l] \arrow[u, hook, "\varphi"'] \arrow[ur] &
\end{tikzcd}$$